ASPECTS OF FLUX COMPACTIFICATION

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I hereby declare that this thesis has not been and will not be, submitted in whole or in part to another University for the award of any other degree.

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SUMMARY

This thesis discusses various features of compactifications of string and M-theory with non-vanishing fluxes on the compact space.

There is a review of the mathematics of special holonomy, its generalisation to G-structures, and relevance for compactification. There is discussion of eleven-dimensional supergravity and M-theory, and a broad exposition of Kaluza-Klein reduction of string- and M-theory and the problems associated with this programme.

After this, there is general discussion of the effective theory produced from the reduction of M-theory on G_2-structure manifolds and then on the degrees of freedom present in the low-energy theory given weak G_2 holonomy.

A new solution to M-theory on manifolds of G_2 holonomy is then presented, with a domain wall in the non-compact space.

Finally, the effective action for massive IIA string theory on manifolds of $SU(3)$ structure is discussed, together with the stabilisation of the moduli for the space $SU(3)/U(1) \times U(1)$ using the $\mathcal{N} = 2 \to \mathcal{N} = 1$ super-Higgs mechanism.

[*160 words*]

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Chapter 1

Introduction

It has long been a goal to unify all known physics. In a modern context, this means the combination of the quantum-field-theoretic approach of high-energy particle physics with the geometrical approach of general relativity. While sensible Lagrangians can be written down that couple gravity with the particle content of the standard model, such a 'naïve quantum gravity' will typically be highly divergent, even at first-loop level, although classical calculations using such setups are possible.

Faced with the non-renormalisability of naïve quantum gravity, there are various responses. One is to undertake some radical re-thinking of our ideas about spacetime, geometry, quantisation and so on. Approaches to quantum gravity that fall into this category are, for example, loop quantum gravity and causal set theory. Another is to investigate whether naïve quantum gravity is in fact non-perturbatively convergent, or to look for some other field-theory technique to allow sensible calculations to be done within the framework of naïve quantum gravity.

String theory sits somewhere between these two approaches. It grew rather organically out of theories of the strong interaction that predated QCD, however attempts to make the theory fully consistent produced strange results. In particular, a fully consistent quantum theory of strings must be supersymmetric and ten-dimensional.

Supersymmetry is a popular extension of the standard model of particle physics, with supergravity being the local extension of supersymmetry. Imposing supersymmetry constrains both the particle content and form of a theory, as well as offering the possibility of non-trivial unification of the gauge and spacetime symmetries. Perhaps most significantly, supergravities are typically far better behaved (although still perturbatively divergent) when their quantisation is attempted. Since string theory is the only known ultra-violet completion of a supergravity, it has been argued that evidence for low-energy supersymmetry at the LHC would provide very strong indirect evidence for string theory, since supergravity is such a natural extension of supersymmetry and string theory is such a natural extension of supergravity.

The presence of extra dimensions, while at first appearing to be in dramatic conflict with observation, in fact has many features to recommend it. It is worth noting that any prediction of the number of dimensions must in some sense be a good thing—before string theory, it seemed that only anthropic reasoning could give a 'prediction' for this quantity. More importantly, the compactification of extra dimensions opens up the possibility of dynamically generating parameters in four dimensions that would otherwise be put in by hand.

The simplest example of a compactification—pure gravity in five dimensions on a circle—gives a good example of this, where the four-dimensional theory contains, in addition to gravity, a gauge field and scalar whose couplings are given in terms of the radius of the circle. Given many extra dimensions and a far richer higher-dimensional theory as we have in string theory, it is not at all implausible that a similar feat will be possible that yields the standard model at low energies.

In broad terms, therefore, this work concerns itself with the dimensional reduction of string theory and also of M-theory, a postulated eleven-dimensional theory uniting the five known string theories. Of course, this topic is rather a large one, and so we will first introduce some background material before proving new results.

In chapter 2 we discuss some of the mathematical results concerning special holonomy and G-structures that will be relevant later. Chapter 3 concerns eleven-dimensional supergravity, its relationship to string theory, the concept of branes and some of its compact solutions. More general concepts and challenges for Kaluza-Klein theory are discussed in chapter 4. The material in these chapters is quite standard, although our presentation of it is new—i.e. is not drawn verbatim from any other source.

The main new results of this work were previously published in [1–3], and are presented in chapters 5, 6, 7 and 8. We now turn to the ideas that motivate each of these calculations in turn. As is mentioned explicitly in the title of this work, their common theme is the inclusion of non-vanishing flux in the compactification background.

1.1 Why flux compactification?

It has long been hoped that low-energy compactifications of string- and M-theory will lead to phenomenological predictions that could be tested experimentally. A major obstacle to achieving this goal is the presence of many light scalars, or 'moduli', which are often flat directions of the four-dimensional theory. These arise as the massless modes of the higher-dimensional matter fields and as gauge-independent variations of the metric on the compact space. The precise values taken by the moduli in the vacuum yield various parameters in the four-dimensional model and therefore act as predictions coming from string theory; if they are flat directions they parameterise a huge vacuum degeneracy.

A pressing concern regarding string theory compactifications, therefore, is the issue of moduli stabilization, which practically means generating a four-dimensional scalar potential with a minimum. One of the ways of inducing a non-trivial classical potential for the low energy fields in the four-dimensional effective theory is through the inclusion of non-vanishing field strengths for the higher-dimensional fields with directions purely in the internal manifold. These are referred to as fluxes and have been used extensively in the literature for the purposes discussed above—see [4–7] for some of the earlier work. It has been known for some time [8–10] and has recently been studied in more detail [11–14], that flux of anti-symmetric tensor fields can, under certain circumstances, be an effective tool to fix the moduli.

A counter-example to this is the case of low-energy M-theory on manifolds of G_2 holonomy. For this scenario the superpotential and Kähler potential, which determine the potential for the scalar fields, were given in [11, 15]. On its own, the potential generated does not fix the moduli, and recent work on moduli fixing for M-theory on G_2 manifolds required non-perturbative contributions [16]. It is, therefore, incorrect to assume that the presence of fluxes is always sufficient to fix moduli, although they may play an important role in the process of doing so.

The presence of background fluxes—when not perturbatively small—in M-theory as well as in string compactifications induces a back-reaction on the underlying geometry so that Ricci-flat manifolds admitting $\mathcal{N} = 1$ supersymmetry in four dimensions are no longer solutions to the equations of motion. In turn the resulting geometry can be described in terms of the concept of G-structures [17–40]. As we will discuss in chapter 2, G-structure manifolds can be classified in terms of their non-vanishing torsion classes.

For M-theory most work is done on G_2-structure manifolds [2, 41–43], although $SU(3)$-structure manifolds have also been considered [32, 37]. In the case of string theory both $SU(3)$- and $SU(2)$-structure manifolds have been considered [35, 39, 44–47]. For a general review of structure manifolds in string and M-theory see [40] and references therein.

From the point of view of phenomenology G-structure manifolds have the advantage that, although they are formally more general than G-holonomy manifolds, they typically have a much simpler field content. This can be thought of as the torsion placing restrictions on the possible metric deformations of the manifold. They also have the feature that, since they are not Ricci-flat, the four dimensional background will not be Minkowski but, at least in the case where some supersymmetry is preserved, anti-de Sitter (AdS).

1.2 M-theory on G_2-structure manifolds

It is well known that M-theory compactified on manifolds with G_2 holonomy leads to four-dimensional effective supergravities with $\mathcal{N} = 1$ supersymmetry [48–50]. M-theory on manifolds with restricted structure group was also studied in [17, 51]. Here it was shown, amongst other things, that if the structure group is exactly G_2 the only supersymmetric solution with non-vanishing fluxes is given by the Freund-Rubin compactification [48] and that the internal manifold must be 'weak G_2'.

A phenomenological advantage of weak G_2 holonomy spaces comes from the consideration of chiral fermions. For typical Kaluza-Klein reductions of M-theory, the spectrum of fermions in four dimensions is in general non-chiral and thus not suitable for particle physics. This is due to a quite general index theorem for smooth seven-dimensional spaces [52] that we discuss further in section 4.3.3.

The resolution of this problem came with the advent of dualities (as discussed in section 3.2) and the realisation that chiral fermions appear at singular conical points in the seven-dimensional manifold [53, 54], together with a mechanism for the cancellation of the appropriate anomalies [55].[1] Since this realisation, there has been considerable activity on the subject of M-theory on G_2 spaces [11, 15, 18, 56, 58–68].

Conical singularities are quite well understood in the non-compact case [69], but so far it has proved difficult to construct compact G_2 holonomy manifolds with point-like singularities. This is partly because the known way in which compact G_2 holonomy manifolds are constructed is by the orbifolding of the seven-torus followed by the blowing up of the orbifold singularities [66, 70, 71], which does not naturally produce point-like singularities.

The known examples of compact manifolds with codimension seven singularities are not in fact manifolds with G_2 holonomy, but weak G_2 manifolds, and it is known [69] that chiral fermions can also emerge from singular weak G_2 manifolds.

[1] Such compactifications turn out to be dual to intersecting brane models in the context of Type IIA string theory, as originally discussed in [56, 57].

These manifolds can be constructed by adding an extra compact dimension to weak-$SU(3)$ spaces to form a 'lemon' with two conical singularities [72]. Although having two singularities presents phenomenological problems, spaces such as these may play a role as simple models involving conical singularities, and motivate further study into general features of weak G_2 manifolds.

Before such specific examples can be considered, it is necessary to consider the general form of the effective theory obtained from the reduction of M-theory on weak G_2 manifolds. In several works it has been proposed that a superpotential depending on the structure is generated, but in most cases the expression of such a superpotential in terms of the low-energy fields and its relevance for moduli stabilisation was not possible to compute due to the lack of understanding of the low-energy degrees of freedom that appear in such compactification.

Compactifications on manifolds with non-trivial G-structure are, in this sense, poorly understood, due to lack of knowledge about their deformation spaces. The purpose of chapter 6 is therefore to clarify some aspects of these compactifications for the particular case of manifolds with weak G_2 holonomy.

One route to understanding G-structure manifolds in string and M-theory compactifications is via dualities [44, 73–82]. In this way it is possible to deduce certain properties of the moduli space of these manifolds. To our knowledge this was only done in [73], where the low energy action was also computed. For such models an explicit expression for the superpotential in terms of the low energy fields can be found [44] and thus it is possible to say something about the dynamics (and in particular moduli stabilisation). The main assumption in these papers was that the fluxes/intrinsic torsion are perturbatively small and that in such a regime the moduli space is similar to the moduli space of some Calabi-Yau manifold. The generalisation of this conjecture for large fluxes/intrinsic torsion is, however, not known.

In a recent work, another conjecture about the deformation space of nearly-Kähler manifolds (known also as manifolds of weak $SU(3)$ holonomy) was made [82]. In this paper it was proposed that the weak-$SU(3)$ conditions impose strong constraints on the space of possible perturbations of such manifolds and only size deformations of these manifolds are allowed.

In the chapter 5 we study some general aspects of M-theory compactifications on manifolds with G_2 structure. By compactifying certain fermionic terms we derive the general form of the superpotential which appears in such compactifications. The formula we obtain generalises in a natural way the result obtained in [15] for just "F_4" fluxes based on the general analysis in [83, 84].

In chapter 6, motivated by the fact that the manifolds with weak $SU(3)$ holonomy are tightly constrained, we study their seven-dimensional relatives, namely manifolds with weak G_2 holonomy. Such manifolds have the property that the intrinsic (con)torsion is a singlet under the structure group G_2 and it turns out that one can infer enough about their internal structure to allow us to derive the low-energy effective action for M-theory compactified on such manifolds. In the end we link with chapter 5 by showing that the general formula for the superpotential derived for any G_2 structure produces the correct result for the manifolds with weak G_2 holonomy, meaning that the potential which is obtained from the compactification can also be obtained from the general superpotential when inserted into the standard $\mathcal{N} = 1$ formula.

In this chapter we go one step further than previous work on weak G_2 compactifications and present the generic form of the low-energy theory obtained in such compactifications. It is important to observe that the four-dimensional ground state in this case is AdS, which together with the presence of a non-trivial flux along the four spacetime dimensions changes the definition of the mass operators for the fields which appear in the low-energy theory.

This should tell us that the appropriate AdS massless modes no longer appear when the fields are expanded in harmonic forms, but in forms $\{\alpha\}$ which satisfy $d\alpha = -\tau \star \alpha$, where τ measures the intrinsic torsion of the manifold with weak G_2 holonomy. That is because these are precisely the variations of the G_2 structure that are compatible with the weak G_2 conditions. Thus one can see in the same way as on manifolds with G_2 holonomy that the modes coming from the matter fields combine with the modes coming from the metric into the complex scalars $a_3 + i\varphi$.

1.3 G_2 domain walls in M-theory

Generally, there are two somewhat complementary ways to approach flux compactifications [85, 86]. Firstly they can be studied using the higher-dimensional theory by computing the (supersymmetric) deformations of the G_2 background due to non-vanishing flux. Typically one expects the flux to deform the G_2 space, introduce warping and modify the external four-dimensional Minkowski space to a domain wall, as in the analogous case for Calabi-Yau manifolds [85–87]. Examples of these domain wall solutions have been studied in [88, 89]. A systematic analysis of such flux backgrounds can be carried out by applying the formalism of G structures to M-theory compactifications.

Alternatively, the problem can be approached from the viewpoint of the four-dimensional effective supergravities that arise from a flux compactification on (undeformed) G_2 spaces. The general structure of such theories, including a formula for the flux superpotential, has been derived in [15]. Due to the presence of the non-trivial superpotential, the simplest solution to these theories is not four-dimensional Minkowski space but, rather, a domain wall.

The main goal of chapter 7 is to analyze G_2 flux compactifications from both viewpoints and discuss the relation between them. On the one hand, we will compute the supersymmetric deformation of the eleven-dimensional G_2 background due to flux. This will be done to linear order in flux, following the logic of the calculation in [87, 90]. We will then consider the associated four-dimensional $\mathcal{N} = 1$ supergravities and find their exact BPS domain wall solutions. It is shown that these four-dimensional BPS domain walls can be viewed as the zero-mode part of the full eleven-dimensional solution. We also demonstrate that the solutions can be supported by either a membrane, located entirely in the external space, or an

M5-brane wrapping an associative three-cycle within the G_2 space.

We consider these results to be phenomenologically relevant in two ways. Although the G_2 domain walls do not respect four-dimensional Poincaré invariance they may still provide a basis for phenomenologically viable compactifications. This is because non-perturbative effects that can be included in the four-dimensional effective theory may yet produce a minimum of the potential [8,12,16,91] and modify the domain wall to a four-dimensional maximally symmetric space. More directly, our solutions represent the simplest way in which a membrane (or a wrapped M5-brane) would appear in a four-dimensional universe if it indeed arises from G_2 compactification of M-theory. In this sense, our results may provide the starting point for an analysis of topological defects in an M-theory universe.

1.4 (massive) IIA string theory on $SU(3)$-structure manifolds

A phenomenologically important feature of $SU(3)$-structure manifolds is that known solutions on these manifolds to ten-dimensional Type IIA and IIB supergravities preserve $\mathcal{N} = 1$ supersymmetry [35, 39, 45–47], rather than $\mathcal{N} = 2$ supersymmetry, since the latter is problematic as a low-energy symmetry due *inter alia* to its lack of chiral representations.

In chapter 8 we will consider compactifications of Romans' massive Type IIA supergravity on manifolds with $SU(3)$ structure. An important advantage of Type IIA theory as opposed to Type IIB is that fluxes alone can generate non-trivial potentials for both complex structure and Kähler moduli, although fully stabilising the moduli has required non-perturbative effects such as instanton corrections [92]. Recently, this has been overcome through the use of orientifolds, where $\mathcal{N} = 1$ AdS solutions with all moduli stabilised have been found [93–95].

As yet, the covariant embedding of the massive IIA theory in the M-theory 'web of dualities' is not known, although it is believed to encode information about the

Type-IIA string theory in D8-brane backgrounds [96]. The massive supergravity theory is also considerably richer than the massless case, and we will not concern ourselves with α' corrections or other 'stringy' effects that would require the full covariant embedding.

We will show that due to the torsion on the internal manifold there are two types of fluxes that are associated with such compactifications: the usual fluxes associated with non-perturbative sources and fluxes originating from vevs of scalar fields. We will then derive the effective low energy $\mathcal{N} = 2$ theory by reducing the gravitino mass terms to obtain the four-dimensional gravitino mass matrix. From this point we will restrict ourselves to the case where the compact space is in a special class of half-flat manifolds shown to be the most general manifolds compatible with the preservation of $\mathcal{N} = 1$ supersymmetry in four dimensions. We will show that for the second type of fluxes the theory can exhibit spontaneous $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ partial supersymmetry breaking in the vacuum, and construct the resulting $\mathcal{N} = 1$ effective theory.

To study the vacua of the theory, we will consider a particular compact space, and show that an $\mathcal{N} = 1$ supersymmetric vacuum exists where all the moduli are stabilised. This stabilisation does not involve the introduction of any non-perturbative effects into the superpotential or the use of orientifolds.

Chapter 2

Mathematical background

We now present some of the more mathematical results that will be useful in the later chapters. We shall treat this material at the ‘physics’ level of formality, although much of it corresponds to relatively recent developments in more formal mathematics.

Throughout, we shall take M to be a generic manifold, with metric g and typical point p.

2.1 Special holonomy

Special holonomy plays an important role in string- and M-theory compactifications, given its role in providing supersymmetric effective theories upon compactification.

2.1.1 Definition of holonomy

The holonomy group is a property of the connection on the manifold. Using the definition from [97], we proceed to define the set of closed curves around the point p

$$\mathcal{C}_p := \{c(t)|0 \leq t \leq 1, c(0) = c(1) = p\} \ , \tag{2.1.1}$$

where c is a curve on M. The holonomy group restricted to the point p is then

$$\mathrm{Hol}(\nabla)|_p := \{P : T_pM \to T_pM | X \in \mathcal{T}_pM, c \in \mathcal{C}_p, \nabla X|_c = PX\} \ , \tag{2.1.2}$$

where T_pM is the tangent space of M at p, and we have used $\nabla X|_c$ to refer to the element of T_pM produced by parallel transport of X around the curve c.

For a connected manifold—which will be considered throughout the rest of this work—the holonomy groups at different points can be shown to be isomorphic, and so we can drop the subscript p in (2.1.2). In general, where M is N-dimensional, $\mathrm{Hol}(\nabla) \subset GL(N, \mathbb{R})$. Orientable manifolds have $\mathrm{Hol}(\nabla) = SL(N, \mathbb{R})$, and metric spaces have $\mathrm{Hol}(\nabla) = SO(N)$.

Where ∇ is the Levi-Civita connection on a space with metric g, the holonomy group $\mathrm{Hol}(\nabla)$ is often written $\mathrm{Hol}(g)$.

2.1.2 Classification of special holonomy manifolds

The possible holonomy groups for a simply connected manifold M of dimension N with irreducible non-symmetric metric g were given in [98], although here we quote from [70]. It turns out that one of the following must hold.

(i) $\mathrm{Hol}(g) = SO(N)$,

(ii) $N = 2n$ for $n \geq 2$, and $\mathrm{Hol}(g) = U(n) \subset SO(N)$,

(iii) $N = 2n$ for $n \geq 2$, and $\mathrm{Hol}(g) = SU(n) \subset SO(N)$,

(iv) $N = 4n$ for $n \geq 2$, and $\mathrm{Hol}(g) = Sp(n) \subset SO(N)$,

(v) $N = 4n$ for $n \geq 2$, and $\mathrm{Hol}(g) = Sp(n)Sp(1) \subset SO(N)$,

(vi) $N = 7$, and $\mathrm{Hol}(g) = G_2 \subset SO(7)$, or

(vii) $N = 8$, and $\mathrm{Hol}(g) = Spin(7) \subset SO(8)$.

All but the first case on this list are said to be *special holonomy*, while the last two are said to be *exceptional holonomy*.

There are known examples of manifolds for each of the cases above. In particular, all Calabi-Yau three-folds have $SU(3)$ holonomy. The discovery of explicit compact spaces of exceptional holonomy was, however, a relatively recent development [99, 100]. The moduli spaces of the G_2 holonomy manifolds constructed using the methods outlined in [70, 99] were considered in [66, 71].

2.2 G-structures

Associated with each holonomy group is a set of globally defined harmonic forms obeying certain algebraic relations. The concept of a G-structure generalises G-holonomy by considering a set of globally defined forms, which we shall call structure forms, with the same algebraic relations but which are not necessarily harmonic.

2.2.1 Generalities

Let us write ϕ_p^i for the i-th globally defined p-form. We can then parameterise the deviation from harmonicity using torsion classes $\mathcal{W}_a$, so that schematically

$$d\phi_p^i = \sum_a \mathcal{W}_a \wedge \phi_{q_{a,i,p}}^{j_{a,i,p}} \; . \tag{2.2.1}$$

To see the relationship between the torsion classes and the usual definition of torsion, we define a metric compatible connection Γ_T such that

$$d^{(T)}\phi_p^i = 0 \; . \tag{2.2.2}$$

The connection Γ_T turns out to be equivalent to the Levi-Civita connection plus a contorsion κ obeying

$$\kappa_{\mu\nu}{}^r = (\Gamma_T)_{[\mu\nu]}^{\;\;r} \; , \tag{2.2.3}$$

where $\mu\nu\ldots$ are spacetime indices. Using (2.2.2), we can write the exterior derivatives of the structure forms as

$$(d\phi_p^i)_{\mu_1\ldots\mu_{p+1}} = (-)^{p+1}(p+1)! \; \kappa_{[\mu_1\mu_2}{}^r(\phi_p^i)_{\mu_3\ldots\mu_{p+1}]r} \; . \tag{2.2.4}$$

The contorsion provides a way of classifying supersymmetric solutions of supergravity theories by considering the structure group G of the manifold as a subgroup of

$SO(N)$. Since in general the contorsion has two antisymmetric indices and one other index, we have that

$$\begin{aligned} \kappa \in \Lambda^1 \otimes \Lambda^2 &\cong \Lambda^1 \otimes so(n) \\ &\cong \Lambda^1 \otimes (g \oplus g^\perp) \\ &\cong \Lambda^1 \otimes g^\perp \ , \end{aligned} \tag{2.2.5}$$

where g is the Lie algebra on G and $g^\perp$ is its complement in $so(N)$. In the last line, we used that the action of g on the structure forms must vanish. We can decompose κ according to the irreducible representations of G in $\Lambda^1 \otimes g^\perp$, which will be spanned by the torsion classes

$$\kappa \in \bigoplus_a \mathcal{W}_a \ . \tag{2.2.6}$$

The precise torsion classes filled by κ will be determined by comparing (2.2.4) and (2.2.1). Spaces of G structure can therefore be classified by which of the torsion classes are filled. In the case where all torsion classes vanish, the space has holonomy G. Note that in this definition we are considering the holonomy with respect to the Levi-Civita connection. When we consider the connection with torsion, it is clear that using the definition (2.1.2) gives $\mathrm{Hol}(\nabla_T) = G$.

2.2.2 Relation to spinors

The G-structures on a manifold can be related to the number of globally defined spinors on the manifold. Suppose we have a set $\{\eta_A\}$ of such spinors. The structure forms can then be constructed from bilinears in those spinors so that

$$(\phi^i_p)_{\mu_1\ldots\mu_p} = \overline{\eta_{A_{i,p}}}\gamma_{\mu_1\ldots\mu_p}\eta_{B_{i,p}} \ , \tag{2.2.7}$$

where the $\{\gamma_\mu\}$ are Dirac matrices on the manifold. Since the connection with torsion that we have defined is still metric-compatible, its action on the vielbeins implicit in the Dirac matrices of (2.2.7) gives zero, and so the condition (2.2.2) is equivalent to

$$D^{(T)}\eta_A = 0 \ , \tag{2.2.8}$$

where we use D for a spinor covariant derivative. (2.2.8) therefore allows us to write the action of the Levi-Civita connection on the globally defined spinors as

$$(D\eta_A)^\mu = \frac{1}{4}\kappa_{\nu\rho}{}^\mu\gamma^{\nu\rho}\eta_A \ . \tag{2.2.9}$$

The form of (2.2.9) is that of a typical Killing spinor equation for a supersymmetric bosonic configuration of a supergravity theory, for example (3.1.5). For vanishing contorsion it becomes the zero-flux condition for a supersymmetric solution as in (3.3.2)—i.e. the existence of a covariantly constant spinor—however there is also the possibility that for non-zero flux and non-vanishing contorsion supersymmetry could still be preserved, as for (3.3.6).

2.2.3 Structure group $SU(3) \subset SO(6)$

As an example of the procedure above, we consider six-dimensional Euclidian manifolds of $SU(3)$ structure, of which the Calabi-Yau is a special case for empty torsion classes. A six-dimensional manifold is said to have $SU(3)$ structure if it admits a nowhere-vanishing two-form J and three-form Ω (with complex conjugate $\overline{\Omega}$) obeying

$$\begin{aligned}
J_m{}^p J_p{}^n &= -\delta_m^n \ , \\
(P_+)_m{}^n\Omega_{npq} &= \Omega_{mpq} \ , \\
(P_-)_m{}^n\Omega_{npq} &= 0 \ , \\
\Omega\wedge\overline{\Omega} &= -\frac{4}{3}iJ\wedge J\wedge J \ , \\
\Omega\wedge J &= 0 \ , \\
\star\Omega &= -i\Omega \ ,
\end{aligned} \tag{2.2.10}$$

where $\star$ denotes the Hodge star and we have defined the usual projectors

$$(P_\pm)_m{}^n := \frac{1}{2}(\delta_m^n \mp iJ_m{}^n) \ . \tag{2.2.11}$$

An alternative definition of an $SU(3)$-structure manifold is a six-dimensional space with an nowhere-vanishing Weyl spinor η_+, with Majorana conjugate η_-, that obey

$$\overline{\eta_+}\eta_+ = \overline{\eta_-}\eta_- = 1 \;, \qquad \overline{\eta_+}\eta_- = \overline{\eta_-}\eta_+ = 0 \;, \tag{2.2.12}$$

and in terms of which we can write the structure forms as

$$\begin{aligned} J_{mn} &:= -i\overline{\eta_+}\gamma_{mn}\eta_+ \;, \\ \Omega_{mnp} &:= \overline{\eta_-}\gamma_{mnp}\eta \;. \end{aligned} \tag{2.2.13}$$

The exterior derivatives of these forms are split into torsion classes as below

$$\begin{aligned} dJ &= -\frac{3}{2}\mathrm{Im}(\mathcal{W}_1\overline{\Omega}) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \;, \\ d\Omega &= \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}_5} \wedge \Omega \;. \end{aligned} \tag{2.2.14}$$

Standard group theory gives

$$\begin{array}{rcccccccccccc} \Lambda^1 \otimes su(3)^\perp & = & (\mathbf{3}+\overline{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} + \overline{\mathbf{3}}) & & & & & & & & & & \\ & = & (\mathbf{1} \oplus \mathbf{1}) & \oplus & (\mathbf{8} \oplus \mathbf{8}) & \oplus & (\mathbf{6}+\overline{\mathbf{6}}) & \oplus & (\mathbf{3}+\overline{\mathbf{3}}) & \oplus & (\mathbf{3}+\overline{\mathbf{3}}) \;, \\ \Rightarrow \kappa & \in & \mathcal{W}_1 & \oplus & \mathcal{W}_2 & \oplus & \mathcal{W}_3 & \oplus & \mathcal{W}_4 & \oplus & \mathcal{W}_5 \;, \end{array}$$

where the torsion classes are aligned to sit underneath the irreducible representations of $SU(3)$ that they are associated with. We now turn to the classification of manifolds of $SU(3)$ structure in terms of their torsion classes.

One important result is that a manifold is only complex when $\mathcal{W}_1 = \mathcal{W}_2 = 0$. Although the proof of this result is non-trivial, note that, for non-vanishing $\mathcal{W}_1$, the relations in (2.2.14) simply cannot be written in holomorphic and anti-holomorphic coordinates, which is possible for all tensors on complex manifolds.

A further class of manifolds have $\mathrm{Re}(\mathcal{W}_1) = \mathrm{Re}(\mathcal{W}_2) = \mathcal{W}_4 = \mathcal{W}_5 = 0$ and are called *half-flat*. Half-flat manifolds with $\mathcal{W}_2 = \mathcal{W}_3 = 0$ are called *nearly Kähler*.

A Calabi-Yau manifold thus has an alternative definition as a manifold of $SU(3)$ structure with completely vanishing torsion classes. Although in this sense, considering $SU(3)$-structure manifolds with non-trivial torsion is more general than the

Calabi-Yau case, in fact the physics that we obtain from such manifolds will often be simpler. For example, it was argued in [82] that nearly Kähler manifolds do not possess any complex structure moduli, and this argument should also apply to half-flat manifolds.

2.2.4 Structure group $G_2 \subset SO(7)$

A seven-dimensional manifold is said to have G_2 structure if it admits a globally defines three-form φ with four-form Hodge dual $\Phi := \star\varphi$. There is no easy set of algebraic relations like (2.2.10) that these forms obey; instead we look at the pullback of φ onto the tangent space, $\overline{\varphi}$. There should be some tangent-space basis $\{e^A\}$ in which this pullback can be written

$$\overline{\varphi} = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \,, \tag{2.2.15}$$

where $e^{A_1 \ldots A_n} := e^{A_1} \wedge \ldots \wedge e^{A_n}$. The four-form pullback can then be written in this basis as

$$\overline{\Phi} = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247} \,. \tag{2.2.16}$$

An alternative definition of a G_2 structure manifold is as a seven-dimensional space with a nowhere-vanishing Majorana spinor η in terms of which we can write the G_2 forms as

$$\varphi^{ABC} = i\overline{\eta}\gamma^{ABC}\eta \,, \qquad \Phi^{ABCD} = -\overline{\eta}\gamma^{ABCD}\eta \,. \tag{2.2.17}$$

The action of Levi-Civita derivatives on the spinor and forms is given in terms of the contorsion κ_{ABC} as

$$D_A^{(T)}\eta = D_A\eta - \frac{1}{4}\kappa_{ABC}\gamma^{BC}\eta = 0 \,, \tag{2.2.18}$$

$$\nabla_A^{(T)}\varphi_{BCD} = \nabla_A\varphi_{BCD} - 3\kappa_{A[B}{}^{E}\varphi_{CD]E} = 0 \,,$$

$$\nabla_A^{(T)}\Phi_{BCDE} = \nabla_A\Phi_{BCDE} + 4\kappa_{A[B}{}^{F}\Phi_{CDE]F} = 0 \,. \tag{2.2.19}$$

The torsion classes are given by

$$\begin{aligned} d\varphi &= \mathcal{W}_1\Phi + \mathcal{W}_2 \wedge \varphi + \mathcal{W}_4\ , \\ d\Phi &= \frac{4}{3}\mathcal{W}_2 \wedge \Phi + \mathcal{W}_3\ . \end{aligned} \tag{2.2.20}$$

The group theoretical result is

$$\begin{array}{rcccccccc} \Lambda^1 \otimes g_2^\perp & = & \mathbf{7}\otimes\mathbf{7} & & & & & & \\ & = & \mathbf{1} & \oplus & \mathbf{7} & \oplus & \mathbf{14} & \oplus & \mathbf{27}\ , \\ \Rightarrow \kappa & \in & \mathcal{W}_1 & \oplus & \mathcal{W}_2 & \oplus & \mathcal{W}_3 & \oplus & \mathcal{W}_4\ , \end{array}$$

where as above, the torsion classes have been aligned to sit underneath the irreducible representations of G_2 that they are associated with.

Also as above, the torsion classes can be used to classify G_2 manifolds. The case where all torsion classes vanish corresponds to G_2 holonomy, while when only $\mathcal{W}_1$ is non-zero, the manifold is said to have *weak* G_2 holonomy.

2.2.5 Relation to the metric

A metric is, in a sense, an $SO(N)$ structure for an N-dimensional manifold, while an orientation is an $SL(N)$ structure. Both of these can be obtained from the kind of forms discussed above as below

$$\begin{aligned} g_{mn} &= (\det(s))^{-1/8} s_{mn} \text{ for} \\ s_{mn} &= -\frac{1}{64}(\Omega_{mpq}\overline{\Omega}_{nrs} + \Omega_{npq}\overline{\Omega}_{mrs})J_{tu}\hat{\epsilon}^{pqrstu}\ , \\ \epsilon_{mnpqrs} &= -15 J_{[mn}J_{pq}J_{rs]} = -\frac{5}{2} i\Omega_{[mnp}\overline{\Omega}_{qrs]}\ , \end{aligned} \tag{2.2.21}$$

$$\begin{aligned} g_{AB} &= (\det(t))^{-1/9}\, t_{AB} \text{ for} \\ t_{AB} &= \frac{1}{144}\varphi_{ACD}\varphi_{BEF}\varphi_{GHI}\hat{\epsilon}^{CDEFGHI}\ , \\ \epsilon_{ABCDEFG} &= 5\varphi_{[ABC}\Phi_{DEFG]}\ . \end{aligned} \tag{2.2.22}$$

Variations of the metric can therefore be linked to variations of the structure forms in each case, giving

$$\begin{aligned}\delta g_{mn} &= -\frac{1}{8}(\delta\Omega)_{(m}{}^{pq}\overline{\Omega}_{n)pq} - \frac{1}{8}(\delta\overline{\Omega})_{(m}{}^{pq}\Omega_{n)pq} - (\delta J)_{t(m}J^{t}{}_{n)} \\ &\quad + \left[\frac{1}{64}(\delta\Omega)\lrcorner\overline{\Omega} + \frac{1}{64}(\delta\overline{\Omega})\lrcorner\Omega - \frac{1}{8}(\delta J)\lrcorner J\right] g_{mn}\end{aligned} \tag{2.2.23}$$

for the $SU(3)$-structure. In this form, calculation is rather difficult, however by using the fact that $P_+ + P_- = \mathbb{1}$, we can obtain some of the calculational convenience of holomorphic and anti-holomorphic coordinates by acting on (2.2.23) with projectors to give

$$\begin{aligned}(P_+)_m{}^p(P_+)_n{}^q\delta g_{pq} &= -\frac{1}{8}\delta\overline{\Omega}_p{}^{qr}(P_+)_{(m}{}^p\Omega_{n)qr}\ , \\ (P_-)_m{}^p(P_-)_n{}^q\delta g_{pq} &= -\frac{1}{8}\delta\Omega_p{}^{qr}(P_-)_{(m}{}^p\overline{\Omega}_{n)qr}\ , \\ \left[(P_+)_m{}^p(P_-)_n{}^q + (P_-)_m{}^p(P_+)_n{}^q\right]\delta g_{pq} &= -\frac{1}{8}\delta\Omega_p{}^{qr}(P_+)_{(m}{}^p\overline{\Omega}_{n)qr} - \frac{1}{8}\delta\overline{\Omega}_p{}^{qr}(P_-)_{(m}{}^p\Omega_{n)qr} \\ -\delta J_{p(m}J_{n)}{}^p &+ \left[\frac{1}{64}(\delta\Omega)\lrcorner\overline{\Omega} + \frac{1}{64}(\delta\overline{\Omega})\lrcorner\Omega - \frac{1}{8}(\delta J)\lrcorner J\right] g_{mn}\ .\end{aligned} \tag{2.2.24}$$

We will refer to the variations of J and Ω as Kähler and complex structure deformations respectively.

For the G_2 case, variations are given by

$$\delta g_{AB} = \frac{1}{2}\varphi_{(A}{}^{CD}\delta\varphi_{B)CD} - \frac{1}{18}(\varphi\lrcorner\delta\varphi)g_{AB}\ . \tag{2.2.25}$$

All relevant conventions are given in the appendix.

Chapter 3

Eleven-dimensional supergravity

Much of our analysis in this work will start from eleven-dimensional supergravity. We outline in this chapter the essential technical details of this theory as well as the reasons for studying it.

3.1 Action and symmetries

The eleven-dimensional supergravity was first presented in [101] and contains the following fields: the three-form $\hat{A}_3$ with field strength $\hat{F}_4 = d\hat{A}_3$, the gravitino Ψ and the graviton g. Using the indices $I, J \ldots$ to run over eleven-dimensional spacetime indices, its action can be written, up to four-fermion terms which we ignore, as

$$\begin{aligned} S_{11} \;=\; & \frac{1}{2}\int_{M_{11}} \left[\star\hat{R} + \frac{1}{2}\hat{F}_4 \wedge \star\hat{F}_4 + \frac{1}{6}\hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \right] \\ & -\frac{1}{2}\int_{M_{11}} d^{11}X\sqrt{-g}\Big[\overline{\Psi}_I \Gamma^{IJK} D_J \Psi_K + \frac{1}{8}\overline{\Psi}^I \Gamma^{JK}\Psi^L (\hat{F}_4)_{IJKL} \\ & +\frac{1}{96}\overline{\Psi}_I \Gamma^{IJK_1K_2K_3K_4}\Psi_J (\hat{F}_4)_{K_1K_2K_3K_4} \Big] \,, \end{aligned} \tag{3.1.1}$$

where we have used the conventions outlined in the appendices.

Typically, we will also consider configurations for which the fermionic field vanishes, so $\langle \Psi_I \rangle = 0$. In this case, the equations of motion are

$$
\begin{aligned}
d\hat{F}_4 &= 0 \, , && (3.1.2)\\
d^\dagger \hat{F}_4 &= -\frac{1}{2}\hat{F}_4 \wedge \hat{F}_4 \, , && (3.1.3)\\
\hat{R}_{IJ} &= \frac{1}{12}\left[(\hat{F}_4)_{IK_1K_2K_3}(\hat{F}_4)_J{}^{K_1K_2K_3} - \frac{1}{12} g_{IJ}\hat{F}_4 \lrcorner \hat{F}_4\right] \, . && (3.1.4)
\end{aligned}
$$

The supersymmetric transformation for the gravitino is then the only non-trivial supersymmetry transformation and is given by

$$
\delta_\epsilon \Psi_I = D_I \epsilon + \frac{1}{288}\left(\hat{\Gamma}_I{}^{J_1J_2J_3J_4} - 8\delta_I^{J_1}\hat{\Gamma}^{J_2J_3J_4}\right)(\hat{F}_4)_{J_1J_2J_3J_4}\,\epsilon \, , \qquad (3.1.5)
$$

where ϵ is a Majorana spinor parameterising the supersymmetry transformation.

3.2 Eleven-dimensional supergravity and M-theory

Eleven-dimensional supergravity is often referred to as M-theory, particularly in the context of compactifications. Strictly speaking, however, it is just the low-energy limit of M-theory, which should be an ultraviolet completion of eleven-dimensional supergravity that unifies the existing five string theories, and may resolve some other outstanding questions in the field. We will now discuss briefly the evidence for M-theory, placing particular emphasis on the implications that this has for thinking about string phenomenology.

3.2.1 'Web of dualities'

There are five known supersymmetric string theories: Type I, IIA and IIB; Heterotic $E_8 \times E_8$ and Heterotic $SO(32)$. Each of these theories has a low-energy limit given by an appropriate supergravity theory in ten dimensions. The superstring theories are believed to be related to each other by a 'web of dualities'—relations between the theories that transform one into the other—that also include eleven-dimensional supergravity [102]. These dualities are often shown diagrammatically as in figure

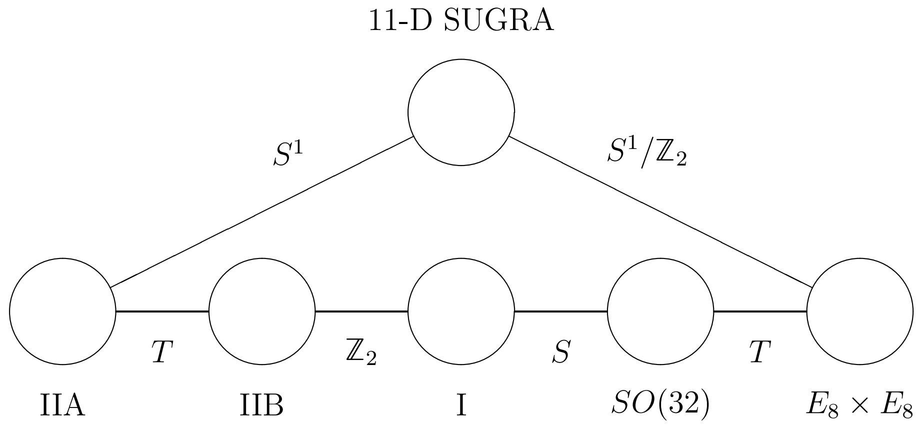

Figure 3.1: M-theory 'web of dualities'.

3.1, which is a schematic representation of the supergravity dualities. Briefly, these consist of the following operations:

S^1 **:** Compactification on a circle. The Kaluza-Klein modes of eleven-dimensional supergravity are dual to solitonic states in IIA supergravity. This is a supergravity duality.

$S^1/\mathbb{Z}_2$ **:** Compactification on an interval. The Kaluza-Klein modes of eleven-dimensional supergravity are dual to solitonic states in Heterotic $E_8 \times E_8$ supergravity. This is a supergravity duality.

T **:** Each string theory is compactified on a circle of radius R. The large-R limit of one theory will be the small-R limit of the other, due to the interchange of string winding and momentum modes, and so this duality is essentially stringy in origin.

$\mathbb{Z}_2$ **:** There is a worldsheet parity symmetry in the IIB string theory, hence $\mathbb{Z}_2$. Projecting out the parity-odd states, and introducing open strings with $SO(32)$ Chan-Paton factors to cancel anomalies, gives the Type I string theory.

S **:** In supergravity terms, the $SO(32)$ Heterotic string and the Type I string are the same. The S duality between them comes from going between strong and weak string coupling, which is given in terms of the dilaton expectation value $\langle\phi\rangle$ as $g_{\mathrm{s}} = e^{\langle\phi\rangle}$. S duality is the transformation $\langle\phi\rangle \to -\langle\phi\rangle$, which is just a field redefinition in supergravity, but changes the regime in which g_{s} is an appropriate expansion parameter in string theory. There is no rigorous proof of this duality in the full string theory case, although there is much indirect evidence.

Although this picture is only fully demonstrated in the case of the supergravity theories, it is conjectured to hold for the full string theories as well. This raises the question—what eleven-dimensional theory would take the place of eleven-dimensional supergravity in the corresponding string-theory picture? This full theory is called M-theory, although as mentioned above, in the context of phenomenology, this term is often used exchangeably with eleven-dimensional supergravity.

3.2.2 Branes

String theory is formulated as a supersymmetric worldsheet theory, which to be consistent must be embedded in a ten-dimensional spacetime. As mentioned above, at low energies different ten-dimensional supergravities are found depending on the boundary conditions imposed on the string worldsheet. These supergravity theories have a range of extended solitonic solutions of various dimensions, as outlined in [103]. In particular, each of the ten-dimensional supergravity theories has a solitonic string state.

At the same time, starting with the embedded worldsheet theories, it is possible to apply two types of open-string boundary conditions: Neumann conditions, for which the string moves freely, and Dirichlet conditions, for which the ends of the string are constrained. Given p Dirichlet conditions, the ends of the string will be constrained to a $(10-p-1)$-dimensional hyperplane. Initially, it was not clear how to interpret Dirichlet boundary conditions, and their lack of Poincaré invariance led to the idea that they were unphysical.

It was with [104] that these hyperplanes were first given their modern interpretation as dynamical objects in their own right, called D-branes. Relics of these dynamical objects can be seen in the solitonic solutions of the supergravity theories, which correspond to 'frozen out' non-perturbative states in the full string theory. Where there is a p-form flux, the $(p-1)$-form field that yields it can be coupled to a $(p-2)$-brane soliton.

Eleven-dimensional supergravity does not have a solitonic string state, but does include a membrane and a dual fivebrane. This suggests that M-theory may be, ultimately, the theory of a supermembrane. Another suggested idea is that of 'brane democracy', meaning that ultimately every supergravity may be the low-energy limit for the quantised worldvolume theory of each of the branes it admits as a solution.

Such hypotheses are hard to test in practice, since for objects of higher dimensions than the string, the quantised worldvolume theory is highly divergent due to the lack of conformal symmetry. Although there are various suggestions for ways around this problem, the most promising of which is matrix theory, this means that the full structure of M-theory is not currently known and so we are restricted to what can be deduced from its low-energy limit, stringy dualities, brane physics and other indirect methods.

Fivebranes require the supergravity action to be written as in [105]. Without quoting the whole result, this involves writing the action in a way that is symmetric under the duality $\hat{A}_3 \leftrightarrow \hat{A}_6$ explained more at the end of chapter 5. The duality symmetric action contains an auxiliary scalar field a, and its coupling to the fivebrane involves a worldvolume two-form B.

3.2.3 Relevance of M-theory

As well as the loftier concerns about the ultimate theory that may be lying behind string theory, which are beyond the scope of this thesis, ideas about duality and branes have a relevance for the attempt to connect string theory with observation—string phenomenology. In particular, it encouraged renewed study of compactifications of both eleven-dimensional supergravity and string theories other than the Heterotic string.

While we deal more comprehensively with the problems of string phenomenology in chapter 4, it is worth noting here that from first principles, there is much to recommend compactifications of eleven-dimensional supergravity. Firstly, since the higher-dimensional theory is the low-energy limit of M-theory, and we expect decompactification at high energies, it is the most natural supergravity to start compactifying. Indeed, the fact that eleven dimensions is maximal from the point of view of supersymmetry is what recommended eleven-dimensional supergravity as the main object of study for some of the initial papers on Kaluza-Klein theories.

Secondly, the Freund-Rubin solution to eleven-dimensional supergravity (presented below) remains one of the most pleasing explanations for the existence of four non-compact dimensions in the context of string- and M-theory. This is made possible by the presence of four-form flux, which can pick out four dimensions. Furthermore, this flux is naturally sourced by the M-theory membrane. Finally, there are several other more speculative reasons, including the possible existence of a 'topological M-theory' (sometimes called Z-theory) with the natural target space of a G_2 manifold.

Despite these *a priori* justifications for the study of compactifications of eleven-dimensional supergravity, there are many practical difficulties involved in its dimensional reduction that we will discuss further in chapter 4. It is to be hoped that these difficulties are not insuperable, and furthermore that this work offers some help in their solution.

3.3 Compactifications of eleven-dimensional supergravity

We shall consider two broad cases of M-theory compactification: the Ricci-flat and the Freund-Rubin. These will be classified according to the number of supercharges that they preserve, which will depend on the holonomy group or structure group of the manifold respectively.

3.3.1 Ricci-flat compactifications

Setting $\hat{F}_4 = 0$ in (3.1.4) tells us that any compactification of the form

$$M_{11} = M_4 \times M_7 \; , \tag{3.3.1}$$

for Ricci-flat M_4 and M_7, will be a solution. In the limit where all of the radii of compactification of M_7 are small, the low-energy theory will be a four-dimensional supergravity, as argued in section 4.2. $\mathcal{N}$ for this supergravity will be given by the number of Killing spinors on M_7, i.e. spinors η satisfying

$$D_A \eta = 0 \; , \tag{3.3.2}$$

where $A, B \ldots$ are seven-dimensional spacetime indices and D is the Levi-Civita spinor covariant derivative. Spinors satisfying (3.3.2) will give vanishing supersymmetric variation for the gravitino at zero flux, from (3.1.5). In fact, for general bosonic configurations the equation $\delta_\epsilon \Psi = 0$ is often called the *Killing spinor equation*, with ϵ called the *Killing spinor*.

From the discussions in chapter 2, it is clear that the number of supersymmetries in the effective theory will be related to the holonomy group of the Levi-Civita connection on the manifold, which here we write $\mathrm{Hol}(M_7)$. This relation takes the form

$$\begin{array}{ccccccccccc} & \mathrm{Hol}(M_7) & = & \mathbb{1} & \subset & SU(2) & \subset & SU(3) & \subset & G_2 & \subset & SO(7)\;, \\ \Rightarrow & \mathcal{N} & = & 8 & & 4 & & 2 & & 1 & & 0\;, \end{array} \tag{3.3.3}$$

where the values of $\mathcal{N}$ are aligned with the appropriate holonomy groups.

3.3.2 Freund-Rubin

The Freund-Rubin solution [106] takes the same product form as (3.3.1), but with the non-vanishing flux

$$\hat{F}_4 = \frac{3}{2}\tau d\mathrm{Vol}(M_4) \ , \tag{3.3.4}$$

where $d\mathrm{Vol}(M_4)$ is the volume element on M_4. From (3.1.4), this means that M_4 must be AdS_4, while M_7 must have constant positive curvature. The Ricci tensors for M_4 and M_7 are in fact

$$R_{\mu\nu} = -\frac{3}{4}\tau^2 g_{\mu\nu} \ , \qquad R_{AB} = \frac{3}{8}\tau^2 g_{AB} \ . \tag{3.3.5}$$

When considering the amount of supersymmetry preserved, the equation (3.3.2) is modified to

$$D_A\eta + \frac{i}{8}\tau\gamma_A\eta = 0 \ . \tag{3.3.6}$$

Following the approach of chapter 2, we can then define a connection Γ_T, which will give $D_A^{(T)}\eta = 0$. The holonomy group of this connection, $\mathrm{Hol}(\Gamma_T)$, will determine the supersymmetries preserved in the four-dimensional theory in the same way that $\mathrm{Hol}(X_7)$ does in (3.3.3) above.

The condition that $\mathrm{Hol}(\Gamma_T) = G$ will be equivalent to saying that M_7 is a G-structure manifold, with torsion classes given as a function of τ. We will later give special attention to the structure group G_2.

Also note that when more general scenarios with additional fluxes, warp factors, non-perturbative effects and so on are considered, the simple relationship between holonomy and supersymmetries will not typically hold.

Chapter 4

Effective four-dimensional theories

Kaluza-Klein reduction involves the production of effective four-dimensional theories from higher-dimensional configurations involving four non-compact dimensions, with the other dimensions compact. In this chapter, we will consider which kind of four-dimensional theories are most commonly attempted to obtain by reduction of higher dimensional theories, as well as the general method of dimensional reduction for field theories.

There is a large body of literature on the topics presented in this chapter. Therefore, rather than citing according to priority, we will often cite works simply because they clearly explain the relevant concepts.

4.1 Four-dimensional supergravities

Four-dimensional supergravities are classified according to their values for $\mathcal{N}$, which is the number of supercharges ϵ their action preserves. In four dimensions such charges will be Majorana or Weyl depending on conventions, and obey

$$\delta_\epsilon S^4_{\rm SUGRA} = 0 \ , \tag{4.1.1}$$

where $S^4_{\rm SUGRA}$ is the four-dimensional supergravity action. In this work, we will consider bosonic configurations, which are those where the fermion expectation values vanish. For such configurations, the gravitino variations will typically take the form

$$\delta_\epsilon \psi_\mu \propto M_{3/2} \gamma_\mu \epsilon \ , \tag{4.1.2}$$

where $M_{3/2}$ is the mass of the gravitino ψ_μ, and γ_μ is a four-dimensional Dirac matrix. Therefore, the number of distinct gravitini in the theory—provided it has been constructed to obey (4.1.1)—will give the value of $\mathcal{N}$ for the action. Furthermore, the value of $\mathcal{N}$ for a given solution to the theory will be given by the number of physically massless gravitini. We will discuss this further in chapter 8.

For now, we turn to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravities, since although $\mathcal{N} = 4$ and $\mathcal{N} = 8$ are possible, they are far more complicated and have phenomenological problems that make them less-favoured theories to be obtained by Kaluza-Klein reduction.

4.1.1 $\mathcal{N} = 1$ supergravity

The full component action of $\mathcal{N} = 1$ supergravity in four dimensions is rather complicated. In supersymmetric theories, however, the fields sit in supermultiplets, which can in this case be combined into superfields. These make the action of four-dimensional $\mathcal{N} = 1$ supergravity rather simpler, giving

$$\begin{aligned} S^4 \ = \ & \int_{M_4} d^4x \sqrt{-g} \int d^2\Theta 2\mathscr{E} \Big[\frac{3}{8} \left(\overline{\mathscr{D}}\,\overline{\mathscr{D}} - 8R \right) \exp \left\{ -\frac{1}{3} \left[\mathcal{K}(\Phi, \overline{\Phi}) + \Gamma(\Phi, \overline{\Phi}, \mathscr{V}) \right] \right\} \\ & + \frac{1}{16 g^2} H_{(ab)}(\Phi) F^{(a)} F^{(b)} + W(\Phi) \Big] + \text{h.c.} \end{aligned} \tag{4.1.3}$$

Each of the symbols above encodes field theoretic information, which we will now explain. Θ represents fermionic coordinates in superspace, with measure $\mathscr{E}$. $\mathscr{D}$ is a superspace covariant derivative, with R the superspace curvature. In components, this curvature contains the graviton $g_{\mu\nu}$, and gravitino ψ_μ. Φ stands for the chiral superfields, which contain spin-0 scalars T^i, as well as spin-$\frac{1}{2}$ fields χ^i, and non-dynamical degrees of freedom, while $\mathscr{V}$ stands for the vector superfields, which contain spin-$\frac{1}{2}$ gauginos λ and spin-1 gauge fields A_μ as well as non-dynamical degrees of freedom. F is the supercovariant field strength of $\mathscr{V}$.

Although the field content as described is common to all four-dimensional $\mathcal{N} = 1$ supergravities, the functionals are used to specify exactly which $\mathcal{N} = 1$ supergravity we are considering. $\mathcal{K}$ is the Kähler potential, Γ is the counterterm, $H_{(ab)}$ is the gauge kinetic funtion and W is the superpotential. These quantities, together with the number of fields, will be what we derive from Kaluza-Klein reduction of higher dimensional theories.

Most often, we will discuss the scalar sector of $\mathcal{N} = 1$ supergravity, which is given by

$$S^4_{\text{scalar}} = \frac{1}{2} \int \left(\sqrt{-g} R + 2 g_{i\bar{\jmath}} \partial_\mu T^i \partial^\mu \overline{T^j} + 2V \right) \ . \tag{4.1.4}$$

$g_{i\bar{\jmath}}$ is the Kähler metric, given in terms of the Kähler potential $\mathcal{K}$ by

$$g_{i\bar{\jmath}} = \frac{\partial^2 \mathcal{K}}{\partial T^i \partial \overline{T^j}} \ . \tag{4.1.5}$$

Field-space indices $i, j, \ldots$ are lowered and raised by $g_{i\bar{\jmath}}$ and its inverse $g^{i\bar{\jmath}}$. The potential V is given by

$$V = e^{\mathcal{K}} \left(g^{i\bar{\jmath}} D_i W D_{\bar{\jmath}} \overline{W} - 3|W|^2 \right) \ , \tag{4.1.6}$$

where W is the superpotential and the Kähler covariant derivative D_i is defined by

$$D_i := \frac{\partial}{\partial T^i} + \left(\frac{\partial \mathcal{K}}{\partial T^i} \right) \ . \tag{4.1.7}$$

4.1.2 $\mathcal{N} = 2$ supergravity

Four-dimensional $\mathcal{N} = 2$ supergravity has a far more complicated Lagrangian than the $\mathcal{N} = 1$ theory, so here we simply outline some of the features of the theory that will be useful later.

As for $\mathcal{N} = 1$ supergravity, the fields sit in supermultiplets, although the $\mathcal{N} = 2$ multiplets typically contain more fields. From [107], these are

Spin	**Field**	**Name**	**Multiplet**
2	$g_{\mu\nu}$	Graviton	
$\frac{3}{2}$	ψ^α, ψ_α	Gravitino doublets	The gravitational multiplet
1	A^0	Graviphoton	
1	A^i	Gauge bosons	
$\frac{1}{2}$	$\lambda^{i,\alpha}, \overline{\lambda}^i_\alpha$	Gaugino doublets	Vector multiplets
0	T^i	Vector multiplet complex scalars	
$\frac{1}{2}$	$\zeta_{a'}, \zeta^{a'}$	Hyperinos	
0	$\xi^a, \tilde{\xi}_a$	Hypermultiplet real scalars	Hypermultiplets.
0	z^a	Hypermultiplet complex scalars	

α is an $SU(2)$ index taking values $1, 2$. The chirality of a fermionic field is positive if its $SU(2)$ index is raised and negative if this index is lowered. i runs from 1 to N_V, a runs from 1 to N_H and a' runs from 1 to $N_H/2$. The raising and lowering of a' determines chirality in the same way as α.

Similarly to the $\mathcal{N} = 1$ theory, the sector of $\mathcal{N} = 2$ supergravity that we will pay most attention to is the scalar sector coming from the vector multiplets and the hypermultiplets. The hypermultiplet scalars $\{\xi^a, \tilde{\xi}_a, z^a\}$ provide coordinates for a quarternionic manifold $\mathcal{M}^Q$ of dimension $4N_H$ while the vector multiplet scalars $\{T^i\}$ provide coordinates for a special Kähler manifold $\mathcal{M}^K$.

As would be expected, it is possible to break $\mathcal{N} = 2 \to \mathcal{N} = 1$ given an appropriate form for the scalar potential. This is generally a rather difficult procedure, in particular finding which fields are made massive and determining the $\mathcal{N} = 1$ quantities W and $\mathcal{K}$ from the $\mathcal{N} = 2$ quantities. There are further phenomenological problems with this mechanism [108], in particular the difficulty of getting chiral fermions in the spontaneously broken theory. Again, we will discuss this further in chapter 8.

4.2 Kaluza-Klein reduction

We turn now to the mechanism by which effective field theories can be derived from higher-dimensional theories, drawing on the approach of [109–111].

We suppose that the higher-dimensional theory is a theory in D spacetime dimensions, with coordinates X^M for $M, N \ldots = 0, \ldots, D-1$. We then look for solutions to this theory with line element

$$ds_D^2 = g_{MN} dX^M dX^N = g_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n \ , \tag{4.2.1}$$

where g_{MN} is the metric on the D-dimensional space M_D, $g_{\mu\nu}$ is the metric on a four-dimensional non-compact space M_4 and g_{mn} is the metric on a $(D-4)$-dimensional compact space K. The coordinates on these last two spaces are x^μ for $\mu, \nu \ldots = 0, 1, 2, 3$ and y^m for $m, n \ldots = 1, \ldots, D-4$ respectively. The line element (4.2.1) thus puts the space X_D in the product form

$$M_D = M_4 \times K \ . \tag{4.2.2}$$

Note that there are generalisations of this procedure to include 'warp factors' dependent on the y^m into the non-compact space. For now, we will consider how various fields behave under the decomposition (4.2.2). In doing this, we will consider simple actions for each field (containing only the kinetic term) and furthermore will not take any field to have a vacuum expectation value (vev). Such an approach brings out the most important features of compactification.

Later on, however, we will consider relaxing some of these assumptions, which turn out not to be appropriate when we consider more general compactification scenarios.

4.2.1 Scalar field

Consider a scalar field ϕ with action

$$\begin{aligned} S_\phi^D &= \int_{M_D} d^D X \sqrt{-g}\, \partial_M \phi \partial^M \phi \\ &= \int_{M_4} d^4 x \sqrt{-g} \left[\mathcal{V} \partial_\mu \phi \partial^\mu \phi + \int_K d^{D-4} y \sqrt{g} \partial_m \phi \partial^m \phi \right] , \end{aligned} \tag{4.2.3}$$

where $\mathcal{V} = \mathrm{Vol}(K)$. This factor of the volume will be removed by a rescaling of the metric, $g_{\mu\nu} \to \mathcal{V}^{-1} g_{\mu\nu}$, as shown in section 4.2.2, and so we concentrate instead on the second term in the second line of (4.2.3). In the limit where the compact space is extremely small, we will be able to ignore terms like $\partial_m \phi$, since non-zero derivatives along the y^m directions will be proportional to the inverse compact radii, and hence the scalar modes associated with them will, from a four-dimensional point of view, look extremely massive.

To see why this should hold, consider the case where K is a circle of radius r. The equation of motion for ϕ is then

$$0 = \nabla_M \nabla^M \phi = \left(\nabla_\mu \nabla^\mu + r^{-2} \partial_y^2 \right) \phi \ . \tag{4.2.4}$$

This equation can be solved by Fourier decomposition of ϕ into

$$\phi(X) = \sum_{p \in \mathbb{Z}} \phi_p(x) e^{ipy} \quad \Rightarrow \quad \left[\nabla_\mu \nabla^\mu - \left(\frac{p}{r} \right)^2 \right] \phi_p = 0 \ . \tag{4.2.5}$$

Clearly, then, the mass of each scalar field mode ϕ_p goes like p/r. So in the limit of small radius, the modes with $p \neq 0$ will be extremely massive, and can be consistently truncated. After substituting (4.2.5) into (4.2.3) and setting $g_{\mu\nu} \to \mathcal{V}^{-1} g_{\mu\nu}, \phi_0 \to \phi, \phi_{p \neq 0} \to 0$ we find the four-dimensional effective action for a scalar field

$$S_\phi^4 = \int_{M_4} d^4 x \sqrt{-g}\, \partial_\mu \phi \partial^\mu \phi \ . \tag{4.2.6}$$

This result generalises to more complicated compact spaces K, with the operator $\nabla_m \nabla^m$ playing the role of a four-dimensional mass that varies inversely with the radii of compactification in the generalisation of (4.2.5). Note that for the scalar field, the reduction is rather simple, and does not depend on any special features of the compact space. Other fields have more involved reductions, but the basic ideas will still hold.

4.2.2 Einstein-Hilbert term

The Einstein-Hilbert term takes the form

$$S^D_{\rm EH} = \int d^D X \sqrt{-g}\, \frac{1}{2}\hat{R}\,, \tag{4.2.7}$$

where $\hat{R}$ is the D-dimensional Ricci scalar. Making arguments similar to those for the scalar field above, we can ignore internal metric derivatives and so using the decomposition (4.2.1) we see that

$$\hat{R} = R + R_K - g^{mn}\nabla^2 g_{mn} - \frac{1}{4} g^{mn} g^{pq} \left(\partial g_{mn} \cdot \partial g_{pq} - 3\partial g_{mp} \cdot \partial g_{nq}\right) \ , \tag{4.2.8}$$

where R is the four-dimensional Ricci scalar, and R_K is the Ricci scalar on K. Inserting the expansion (4.2.8) into (4.2.7), we find that the four-dimensional Ricci scalar picks up a factor of $\mathcal{V}$, which is removed by the Weyl rescaling

$$g_{\mu\nu} \to \mathcal{V}^{-1} g_{\mu\nu}\ . \tag{4.2.9}$$

After performing this rescaling for the Ricci scalar, volume element and external raised indices, we end up with

$$\begin{aligned} S^4_{\rm EH} \;=\; & \frac{1}{2}\int_{M_4} d^4x\sqrt{-g}\Big[R + \mathcal{V}^{-1} R_K + \frac{3}{2}\partial_\mu(\ln\mathcal{V})\partial^\mu(\ln\mathcal{V}) \\ & + \frac{1}{4\mathcal{V}}\int_K d^{D-4}y\sqrt{g}\; g^{mn} g^{pq}\left(\partial_\mu g_{mn}\partial^\mu g_{pq} - \partial_\mu g_{mp}\partial^\mu g_{nq}\right)\Big]\ . \end{aligned} \tag{4.2.10}$$

4.2.3 Form fields

Consider a p-form field A with field strength $F = dA$. This will have action

$$S^D_A = \frac{1}{4}\int_{M_D} d^D X\sqrt{-g}\; F \wedge \star F\ , \tag{4.2.11}$$

and equations of motion

$$dF = d\star F = 0 \qquad \Longleftrightarrow \qquad \Delta_{(D)}F = 0\ . \tag{4.2.12}$$

Now, the D-dimensional Laplacian $\Delta_{(D)}$ splits under the Ansatz (4.2.1) to give

$$\Delta_{(D)} = \Delta_{(4)} + \Delta_{(D-4)}\ . \tag{4.2.13}$$

We then expand F in eigenforms of $\Delta_{(D-4)}$, which is the equivalent of Fourier expansion in the scalar case, so that

$$F = \sum_{q=\max(0,p-D)}^{\min(p,4)} F^i_{(q)} \wedge \omega_{(p-q),i}\ , \tag{4.2.14}$$

where we have chosen $\left\{\omega_{(q),i}\right\}$ as a basis for q-forms on K that are eigenforms of $\Delta_{(D-4)}$. By arguing that the $\omega_{(q),i}$ with non-zero eigenvalues will, in the limit of a small compact space, be extremely massive, we truncate those modes with finite masses, leaving only harmonic forms on the internal space. If we then take $\left\{\pi_{(q),i}\right\}$ as a basis of harmonic q-forms on K, the action (4.2.11) reduces to

$$S^4_A = \frac{1}{4}\sum_{q=\max(0,p-D)}^{\min(p,4)} \int_{M_4} d^4x\sqrt{-g}\; C_{q,ij}F^i_{(q)} \wedge \star F^j_{(q)}\ , \tag{4.2.15}$$

where

$$C_{q,ij} = \int_K d^6y\sqrt{g}\; \pi_{(p-q),i} \wedge \star\pi_{(p-q),j}\ . \tag{4.2.16}$$

Now, since the harmonic forms are in one-one correspondence with the non-trivial cycles on the manifold, it is topological information about K that determines the effective action (4.2.15). In particular, the number of q-form fields will be equal to the qth Betti number on the manifold, and the quantity $C_{q,ij}$ will be given by the intersection numbers of the q-cycles on K.

Note that a scalar field is simply a 0-form, while a vector field is a 1-form, so the analysis here specialises to these cases.

4.2.4 Fermionic fields

For a spinor field Ψ, the action is

$$S_{\Psi}^{D} = \int_{M_D} d^D X \sqrt{-g}\, \overline{\Psi} \Gamma^M D_M \Psi \ , \tag{4.2.17}$$

where Γ^M are the D-dimensional Dirac matrices and D_M is the spinor covariant derivative. The equations of motion are then the standard Dirac equation

$$\Gamma^M D_M \Psi = 0 \ . \tag{4.2.18}$$

The reduction of fermionic fields involves not just splitting the space-time indices, but also the spinor indices. We start by defining

$$\gamma := i \prod_{\mu} \gamma^{\mu} \ , \qquad \gamma_K := i^{D(D-1)/2} \prod_{m} \gamma^{m} \ , \tag{4.2.19}$$

where γ^m are the Dirac matrices on K and γ^μ are the four-dimensional Dirac matrices. γ_K will be proportional to $\mathbb{1}$ for D odd and will be a chirality operator for D even, while γ will be the four-dimensional chirality operator. We can then decompose the D-dimensional Dirac matrices as

$$\Gamma^{\mu} = \gamma^{\mu} \otimes \gamma_K \ , \qquad \Gamma^{m} = \gamma \otimes \gamma^{m} \ . \tag{4.2.20}$$

There is an additional subtlety to do with the Weyl rescaling (4.2.9). Since the Dirac matrices are defined in terms of the metric, when we rescale the metric we should also rescale the Dirac matrices, i.e.

$$\{\gamma_{\mu}, \gamma_{\nu}\} = g_{\mu\nu} \to \mathcal{V}^{-1} g_{\mu\nu} \quad \Rightarrow \quad \gamma_{\mu} \to \mathcal{V}^{-1/2} \gamma_{\mu} \ . \tag{4.2.21}$$

Taking η_i as a basis of spinors on K, and ψ^i as a set of four-dimensional spinors, we write

$$\Psi(X) = N \psi^i(x) \otimes \eta_i(y) \ , \tag{4.2.22}$$

where N is a normalisation factor, following the metric Ansatz (4.2.1). The Dirac equation then reads

$$\left(\gamma^{\mu} D_{\mu} \psi^i\right) \otimes \gamma_K \eta_i + \gamma \psi^i \otimes \left(\gamma^m D_m \eta_i\right) \ . \tag{4.2.23}$$

Now, using arguments similar to the above, if we choose the η_i to be eigenspinors of the Dirac operator $\gamma^m D_m$, then only the ones with zero eigenvalue will appear in the effective theory for the limit of small K.

There are, however, subtleties in constructing the effective action, due to the fact that Dirac spinors are not in general irreducible spinor representations, and have to be decomposed. For some numbers of dimensions, the irreducible representations can be Majorana, in some they can be Weyl and in others they can be Majorana-Weyl. Although this may affect how we count spinor fields, for the moment we simply take ξ^i to be a basis for eigenspinors of the Dirac operator with zero eigenvalue, and write the effective action as

$$S^4_\psi = \int_{M_4} d^4x\sqrt{-g}\; c_{ij}\overline{\psi^i}\gamma^\mu D_\mu \psi^j \ , \tag{4.2.24}$$

where

$$c_{ij} = \mathcal{V}^{1/2}|N|^2 \int_K d^6y\sqrt{g}\;\overline{\xi_i}\xi_j \ . \tag{4.2.25}$$

N should then be chosen to put the kinetic terms in (4.2.24) in canonical form, which will typically mean that $N \propto \mathcal{V}^{-1/4}$. Unlike the form fields above, the zero eigenspinors of the Dirac operator are not related to simple topological quantities like the Betti numbers, but are instead given by special holonomy (or more generally G-structure) considerations.

For Rarita-Schwinger fields like gravitini, the procedure is very similar to the above, together with the reduction of the spacetime index. Such a field Ψ_M will yield both a four-dimensional Rarita-Schwinger field from the terms in Ψ_μ and D spin-$\frac{1}{2}$ fields from the terms in Ψ_m.

4.3 Some phenomenological problems

Physically, it is hoped that Kaluza-Klein reduction can reproduce all features of currently observed four-dimensional physics—i.e. the standard model—together with extensions that have not been ruled out by data, such as TeV-scale supersymmetry.

The attraction of such a reduction would be that fundamental quantities in the four-dimensional theory such as coupling constants, particle content and even quantities like the cosmological constant can be related to features of the compact space and higher-dimensional theory.

Typically, the higher-dimensional theory will be simpler, and may be derived from still more fundamental principles. Furthermore, the compactification of the internal space (or decompactification of the external space) should be a dynamical process, leading to an explanation of the multiple parameters of the standard model.

Although the discussion above has been general, in that it has not assumed any particular higher-dimensional theory, it is really in the context of string and M-theory that the Kaluza-Klein approach is most explanatory. This is because the theories in question should provide quantum theories of gravity, and also have enough dimensions and sufficient field content to be 'rich' enough to encode the standard model given very few (if any) fundamental parameters in the higher dimensional theory.

We now turn to some of the challenges facing the compactification of string and M-theory, a field often called string phenomenology.

4.3.1 Particle content

As mentioned above, a successful Kaluza-Klein theory should be able to reproduce the particle content of the standard model. In particular, this means that there should be three generations of quarks and leptons with the appropriate Yukawa couplings to the Higgs sector to generate values within the experimental bounds for the CKM mass matrix for quarks, and the MNS mass matrix for leptons, after electroweak symmetry breaking.

For $E_8 \times E_8$ Heterotic string theory compactification on six-dimensional spaces, as outlined in [110], the number of generations of fermions N_{g} is given by

$$N_{\mathrm{g}} = |\chi(K)/2| \ , \tag{4.3.1}$$

where $\chi(K)$ is a topological property of the compact space K called the Euler number, which is related to the Betti numbers by

$$\chi(K) = \sum_p (-1)^p b_p(K) \ , \tag{4.3.2}$$

where $b_p(K)$ is the p-th Betti number of K. Although other compactification scenarios do not necessarily have such a simple relationship between the number of generations and topological properties, arriving at the correct number of generations should still involve some special subclass of the type of space under discussion.

What is more problematic, however, is finding exactly the correct Yukawa couplings. These can most easily be obtained when the compact space is an orbifold, and the mass hierarchies within fermion families can be given by the distances between the orbifold fixed points. As yet, however, there is no compactification that gives a set of Yukawa couplings that is completely consistent with standard model observations.

4.3.2 Gauge group

The gauge group for the standard model is $SU(3) \times SU(2) \times U(1)_Y$, which breaks to $SU(3) \times U(1)_{\rm EM}$ via the Higgs mechanism. There are many possible groups that could themselves spontaneously break to the standard model—the so-called GUT (Grand Unified Theory) groups. [112] gives a flavour for just how many possibilities there are. Note that non-supersymmetric GUT theories actually suffer from many of the same problems as the non-supersymmetric standard model, in particular the hierarchy problem. This does not apply, however, to supersymmetric GUTs, which are still popular 'bottom up' extensions to the standard model.

Of course, as well as obtaining the correct gauge group, compactification should yield the observed gauge couplings at the standard model energy scale, or GUT gauge couplings that can spontaneously break to the standard model values at lower energy.

Clearly, the groups $SO(32)$ and $E_8 \times E_8$ associated with the Heterotic string are large enough to contain the standard model gauge group. If anything, these groups are larger than would typically be postulated for phenomenological reasons. The other string theories and M-theory, however, do not have fundamental gauge groups and so some other approach is required to generate non-Abelian gauge groups in four dimensions.

One route to a larger gauge group is from the isometry group of the compact manifold, which is the group of transformations leaving its metric invariant. Infinitesimally these transformations can be written as

$$y^m \to y^m + \delta^a \xi_a^m \ , \tag{4.3.3}$$

where a are Lie algebra indices δ^a are parameters of the transformation and $\{\xi_a\}$ are the Killing vectors of the space K. These obey the relation

$$\xi_b^n \partial_n \xi_c^m - \xi_c^n \partial_n \xi_b^m = -C^a{}_{bc} \xi_a^m \ , \tag{4.3.4}$$

where $C^a{}_{bc}$ are the structure constants of the isometry group G. Defining $\mathcal{A}_m^\mu := \xi_a^m A_\mu^a$ shows how fluctuations parameterised by $\mathcal{A}_m^\mu$ around the ground-state metric given in (4.2.1) give rise to the vector field A_μ^a with gauge group G via

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{A}_\mu^m \mathcal{A}_{m,\nu} \ , \qquad g_{m\mu} \to \mathcal{A}_{m,\mu} \ , \qquad g_{mn} \to g_{mn} \ . \tag{4.3.5}$$

Gauge transformations are therefore generated by isometries of K. While there are manifolds that have isometry group $SU(3) \times SU(2) \times U(1)$ or larger—typically constructed as cosets G/H for some $H \subset G$—Calabi-Yau and G_2-holonomy spaces typically have trivial isometry group and so cannot give non-Abelian gauge groups from metric variations.

Another possibility is to introduce singularities into the compact space [69]. These are singularities that look locally like $\mathbb{C}^2/\Gamma$ for some finite subgroup $\Gamma \subset SU(2)$. For each such non-trivial Γ there is a corresponding Dynkin diagram from $A_{n\geq 1}$, $D_{n\geq 4}$ or $E_{n=6,7,8}$ [70], which is the reason that these are often called A-D-E singularities. It turns out that such singularities give gauge group according to the appropriate Dynkin diagram, i.e. $G = SU(N \geq 2)$, $SO(2N \geq 8)$ or $E_{N=6,7,8}$ respectively. As an example, $\Gamma = \mathbb{Z}_2 \Rightarrow G = SU(2)$.

Finally, motivated by M-theory considerations (see section 3.2), the worldvolume action for N coincident branes has $U(N)$ symmetry [113, 114]. This enhanced symmetry, rather than the $U(1)^N$ that one would get from separated branes, comes from the massless limit of string modes stretching between the branes. Where the branes fill the external space, either because they are three-branes or because they are $(p > 3)$-branes wrapped on calibrated $(p - 3)$-dimensional submanifolds of the compact space, then the presence of branes 'frozen' into the low-energy theory can give $U(N)$ symmetry.

Of course, these possibilities for the generation of gauge groups are not mutually exclusive: one could dimensionally reduce a theory with a fundamental gauge group, on a manifold with non-trivial isometry group and A-D-E singularities in the presence of D-branes. This reflects in part the shift in emphasis of string phenomenology from the search for the standard model in string theory to the realisation that there are likely to be many ways of generating the standard model, and concerns about what process selects the appropriate one.

4.3.3 Chiral fermions

It has been known for some time that the weak interaction maximally violates parity. In field-theoretic terms, this means separating fermionic fields into different chiralities, since in four dimensions Dirac spinors are not irreducible representations and so must be separated into Weyl (or Majorana) components. Each of the two chiralities has different couplings to the weak force, hence the violation of parity.

When a higher dimensional theory is compactified, the four-dimensional chiralities of the fermions are linked to the chiralities of their counterparts on the compact space. To see this, we continue the discussion (and notation) of section 4.2.4, making use of [52, 97, 110]. We start by defining

$$\begin{aligned}\Gamma_4 &:= i\prod_\mu \Gamma^\mu = \gamma\otimes\mathbb{1}\ ,\\ \Gamma_K &:= i^{D(D-1)/2}\prod_m \Gamma^m = \begin{cases}\gamma\otimes\mathbb{1} & \text{for } D \text{ odd,}\\ \mathbb{1}\otimes\gamma_K & \text{for } D \text{ even.}\end{cases}\end{aligned} \tag{4.3.6}$$

This gives the expected result for the higher-dimensional parity operator

$$\Gamma = \Gamma_4\Gamma_K = \begin{cases}\mathbb{1}\otimes\mathbb{1} & \text{for } D \text{ odd,}\\ \gamma\otimes\gamma_K & \text{for } D \text{ even.}\end{cases} \tag{4.3.7}$$

Note that although the case of odd D may appear trivial, we will keep the discussion below general. Now, as mentioned before, the higher-dimensional fermion field Ψ will not in general be a Dirac field, but will be in one of the irreducible representations obtained using the chirality operator, so that $\Gamma\Psi = \pm\Psi$. Such a condition clearly gives that $\Gamma_4\Psi = \pm\Gamma_K\Psi$, and so links external and internal parity.

One important quantity to consider using this relation is the index of the Dirac operator $\gamma^\mu D_\mu$. This operator splits into two operators $\mathcal{D}$ and $\mathcal{D}^\dagger$, which are non-vanishing on positive and negative chirality fermions respectively. The index is then given by

$$\begin{aligned}\text{index}(\gamma^\mu D_\mu) &= \dim(\ker(\mathcal{D})) - \dim(\ker(\mathcal{D}^\dagger))\\ &= \nu_+ - \nu_-\ ,\end{aligned} \tag{4.3.8}$$

where $\nu_\pm$ is the number of zero-mass modes with chirality ± 1. Clearly, we will want this quantity to be non-zero to give chiral fermions in four dimensions. Using the relationship between four-dimensional and D-dimensional chirality for positive $(D+4)$-dimensional chirality, we see that

$$\text{index}(\gamma^\mu D_\mu) = \text{index}(\gamma^m D_m)\ . \tag{4.3.9}$$

Together with the Atiyah-Singer index theorem, this tells us that

$$\nu_+ - \nu_- = \int_K \hat{A}(TK)|_{\mathcal{V}}\ , \tag{4.3.10}$$

where $\hat{A}$ is the Dirac genus and TK is K's tangent space. From its definition, $\hat{A}$ contains only (0 mod 4)-forms, giving the result

$$\dim(K) \neq 0 \bmod 4 \;\Rightarrow\; \nu_+ - \nu_- = 0 \; . \tag{4.3.11}$$

This creates a *prima facie* problem for any string or M-theory compactification, where the values for $\dim(K)$ are $D = 6, 7$ respectively, however there are several ways around this result.

Where there is a fundamental gauge field with group G, as in the Heterotic string, the Dirac operator picks up a term in the gauge field, changing the result (4.3.10) from the index theorem to

$$\nu_+ - \nu_- = \int_K \hat{A}(TK)\mathrm{ch}(E)|_{\mathcal{V}} \; , \tag{4.3.12}$$

where $\mathrm{ch}(E)$ is the total Chern character of E, which is the associated vector bundle for the principal bundle $P(K, G)$. Of course, the result (4.3.12) still places constraints on which gauge-group/manifold combinations will provide chiral fermions in four dimensions.

As mentioned above, the other string theories and M-theory do not have fundamental gauge groups, and so again some alternative methods must be found to obtain chiral fermions in four dimensions.

Analogously to the A-D-E singularities discussed above, it is also possible to introduce conical singularities into the compact space K to give chiral fermions. These are singularities with codimension $\dim(K)$, and hence involve essentially four-dimensional physics, with chiral supermultiplets sitting at each singularity. Care must also be taken to ensure that the chiral and other anomalies produced at these singularities are cancelled by inflow from the bulk theory as in [54, 55, 115, 116].

Also, branes can produce chiral fermions when they intersect at non-trivial angles. In particular, generically intersecting stacks of D6-branes produce a chiral fermion at the intersection charged under the $U(M) \times U(N)$ obtained from intersecting a stack of M and a stack of N D6-branes [117].

4.3.4 Moduli stabilisation and supersymmetry breaking

Typically, when a theory is compactified there are a large number of four-dimensional scalar fields derived from the purely internal components of bosonic fields in the higher-dimensional theory. For the case of $\mathcal{N} = 1$ supersymmetry, the potential for these 'moduli' fields is given by (4.1.6), with the superpotential W and Kähler potential $\mathcal{K}$ given by features of both the compact space and the fields on it.

The usual reason for giving special attention to $\mathcal{N} = 1$ supersymmetry in this context—apart from its other phenomenological benefits—is that for higher supersymmetries, the possible contributions to the scalar potential are in some sense 'special cases' of $\mathcal{N} = 1$ contributions, while for fully broken supersymmetry there are so many possible potentials that any systematic discussion of them is almost impossible.

For string and M-theory compactifications, the typical form of the Kähler potential is

$$\mathcal{K} \propto \ln \mathcal{V} \ , \tag{4.3.13}$$

where $\mathcal{V}$ is the volume of the compact space and the constant of proportionality is an integer given by the underlying theory. Where T^i are the scalar components of the chiral superfields, $\mathcal{K}$ is a function of $(T^i + \overline{T^i})$ as would be expected, although this function can often be quite complicated.

The superpotential then usually takes contributions from three effects—gaugino condensation, instantons and fluxes. The first two of these are non-perturbative in origin, and are usually included in the superpotential in the form

$$W_{\rm np} = \sum_i k_i e^{-\alpha_i T^i} \ , \tag{4.3.14}$$

where k_i, α_i are constants. Gaugino condensates are formed when there is a non-Abelian gauge group present, while instantons are string or brane effects from Euclidian worldvolumes wrapping calibrated cycles on the manifold.

Flux terms arise when the p-form fluxes take non-vanishing values on the internal space. Although they are perturbative in the sense of appearing as classical expectation values, they should ultimately be sourced by branes, which are really non-perturbative objects. The rough form for such terms is

$$W_{\text{flux}} = \sum_{i,p} \int_K F^i_{(D-p)} \wedge T^i_p \ , \tag{4.3.15}$$

where F^i_p is the i-th p-form flux and T^i_p is the superfield whose real part is ϕ^i_p, the i-th globally defined p-form on the manifold obeying G-structure relations as in chapter 2.

There are, of course, other terms that can contribute to the superpotential, most importantly from space-filling D3/$\overline{\text{D3}}$-branes or wrapped higher-dimensional branes. As well as contributing particle content as mentioned above, these will typically alter the Kähler potential and superpotential. We will not concern ourselves here or elsewhere in this work with the exact form of such terms, whose form can be quite heterogeneous depending on the precise way in which branes are introduced to the theory.

The problem of moduli stabilisation is, therefore, of trying to generate potentials for the scalar fields T^i with stable local (and preferably global) minima. For some time this was seen as an extremely difficult problem, however recently models have proliferated with stable minima, particularly with the advent of flux compactifications.

Assuming that the moduli can be stabilised at a supersymmetric minimum, the task remains of breaking to a non-supersymmetric theory, ideally at a mass scale suitable for solving the standard model hierarchy problem. Of course, there is always the possibility that the moduli could be stabilised only at a non-supersymmetric vacuum, however in that case it will be harder to obtain the benefits of low-energy supersymmetry. It is also practically easier to search what can be quite complicated potentials for supersymmetric than non-supersymmetric minima, as can be seen from the conditions

$$D_i W = 0 \quad \Rightarrow \quad \text{SUSY minimum;} \tag{4.3.16}$$

$$\partial_i V = 0, D_i W \neq 0 \quad \Rightarrow \quad \cancel{\text{SUSY}} \text{ extremum.} \tag{4.3.17}$$

Here W is the superpotential, V is the potential, the scalar directions are labelled with i, and D_i is the Kähler covariant derivative. As well as being significantly more complicated, the condition (4.3.17) also leaves open the possibility of the extremum being a maximum in some of the directions, while supersymmetry ensures (physical) minimality.

Therefore, a common desideratum for scalar potentials from string- and M-theory compactifications is to give supersymmetric minima, with supersymmetry softly broken at energies much below the Kaluza-Klein scale, although of course this is more easily said than done.

4.3.5 Cosmology

A variety of problems with the 'standard big bang' model of cosmology have led to the widespread (although by no means universal) acceptance of inflationary cosmology, briefly reviewed in [118]. This involves a period in the early universe when the universe's scale factor $a(t)$ has positive definite second derivative with respect to time, which in the FRW setup means

$$\ddot{a} > 0 \quad \Leftrightarrow \quad \rho + 3p < 0 \; . \tag{4.3.18}$$

Assuming positive energy density ρ, this means that inflation happens during a period of negative pressure. The simplest way to implement this is with a scalar field ϕ—the inflaton—which obeys

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \; , \qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \; . \tag{4.3.19}$$

So for a scalar field, (4.3.18) will hold when the potential $V(\phi)$ dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$, a condition often called *slow roll*.

Of course, there are many other ways to implement (4.3.18) than with a scalar field coupled to gravity, and as mentioned some of the problems that inflation was designed to solve are also resolved by alternative theories. Given the successes of inflationary models of density perturbations in fitting data from WMAP, however, it is more common to try to embed inflation in string theory than to come up with a full alternative.

Typically, then, it is hoped that the scalar potential generated by Kaluza-Klein reduction will contain a direction that can behave like the inflaton. This means that we hope that the potential will be suitably flat on the approach to its minimum.

There are, however, deeper problems than just the flatness of the potential. It is a general result that supergravity compactifications cannot yield de Sitter (dS) backgrounds in four dimensions [119]. These are relevant firstly because the period of inflation is well described by a dS phase in the universe's history, and secondly because the current best fits of cosmological data involve a positive cosmological constant which, although it is small compared to the Planck energy, still makes up around 70% of the universe's energy density, although there may be other explanations for this 'dark energy'.

The no-go theorems on de Sitter vacua can be avoided by using non-perturbative effects, however until recently all vacua found had the four-dimensional space as either Minkowski or Anti-de Sitter (AdS). This situation changed with the work of [12], often called the KKLT scenario, mainly due to the inclusion of effects arising from space-filling $\overline{\mathrm{D3}}$-branes, which in the context of their compactification shift the potential by

$$V_{\overline{\mathrm{D3}}} = \frac{D}{\sigma^3} \,, \tag{4.3.20}$$

where D is a (positive) factor depending on *inter alia* the number of $\overline{\mathrm{D3}}$-branes, and σ is the volume modulus for the Calabi-Yau space upon which they compactify the Type IIB string theory.

As well as trying to obtain empirically adequate results for inflation and vacuum energy, there are many other requirements upon string compactifications coming from cosmology. One of these is the provision of a candidate for 'dark matter', which makes up around 30% of the universe's energy density, completely dominating over ordinary baryonic matter. Viewed in the light of this evidence, the 'exotic' particles often yielded by string- and M-theory compactifications may turn out to be an advantage. If the lightest supersymmetric particle is stable, supersymmetry alone may solve this problem, as may low-lying Kaluza-Klein modes, although either of these would not necessarily provide direct support for string theory if observed.

Despite the fact that WMAP has effectively ruled out topological defects as the dominant source of density perturbations in the early universe, there has been renewed interest in them as a sub-dominant source of perturbations (for the case of cosmic strings see [120]). Therefore, generation of topological defects at energy densities not ruled out by measurements may be either a positive or negative feature of a given string- and M-theory compactification depending on how the observations turn out.

4.3.6 Prospects for Kaluza-Klein theory

Solving any individual one of the problems above is a highly complex task; solving all at once may seem impossible. For example, decades of work on compactifications of the Heterotic string have culminated in the 'Heterotic Standard Model' result [121], which contains the correct particle content and gauge group, but since the moduli are not stabilised the Yukawa couplings cannot be compared to experiment.

On the other hand, the KKLT scenario inspired much work involving stable moduli, inflation or even the possibility of phenomenologically viable cosmic strings [122], however the particle content and gauge group of the relevant four-dimensional theories are typically far from the standard model or even quite exotic extensions of it.

This should not, however, be cause for desperation. Thinking about M-theory (see section 3.2) it is more common to see the different areas of string theory as linked, so that a problem that is easy to solve in one 'corner' of M-theory may be dual to a less-easily constructed setup in some other corner. Also, the addition of new 'ingredients' such as fluxes, branes and singularities provides the tools for the kinds of setup that are likely to be needed for full phenomenological viability.

The particular focus of this work is the role of fluxes in string- and M-theory phenomenology. We are more concerned with the scalar potentials that will be obtained, for use in moduli stabilisation and cosmology, than will realistic particle content. Still, we hope not to lose sight of the overall goal: the dimensional reduction of a quantum theory of gravity, that at low energy captures all currently known physics.

Chapter 5

M-theory on G_2 structure manifolds

Compactification on manifolds of G_2 structure is associated with giving $\mathcal{N} = 1$ supersymmetry in four dimensions. As discussed in chapters 2 and 3, G_2 structure manifolds have exactly one Killing spinor. In fact, it has been shown that given the assumption of a Poincaré-invariant external space, the only possibilities for $\mathcal{N} = 1$ given G_2 structure are the direct product of flat space and a manifold of G_2 holonomy, and the direct product of AdS_4 with a weak G_2 manifold [17]. The assumption of Poincaré invariance was relaxed in [123].

Nevertheless, in this section we will not specialise to either of these cases, but will consider the case of general $\mathcal{N} = 1$ compactification of M-theory on G_2 structure manifolds. In section 5.1, we outline some mathematical properties of G_2, while in section 5.2 we construct the $\mathcal{N} = 1$ effective action following compactification.

5.1 Some properties of G_2

We shall now state some results for manifolds of G_2 structure, in part following [18, 70]. We pay particular attention to those features of G_2 that are of use in deriving the results we arrive at elsewhere in this work.

5.1.1 G_2 Lie Algebra

In this section we define G_2 as a subgroup of $SO(7)$ by defining the G_2 3-form φ. We also decompose the Lie algebra of $SO(7)$ into a part in G_2 and a perpendicular part. $SO(7)$ and its Lie algebra are given by

$$\begin{aligned} SO(7) &:= \left\{ O \in Gl(7,\mathbb{R}) \,\middle|\, O = O^T \ \&\ \det(O) = 1 \right\} , \\ \Rightarrow \mathscr{L}(SO(7)) &= \left\{ T \,\middle|\, T^T = -T \right\} \\ &= \operatorname{Span}\{S_{AB}\} , \end{aligned} \tag{5.1.1}$$

where the basis of 21 generator matrices is

$$(S_{AB})^C{}_D := \delta^C_A \delta_{BD} - \delta_{AD}\delta^C_B . \tag{5.1.2}$$

G_2 is then defined as the subgroup of $SO(7)$ whose action preserves the 3-form φ given in (2.2.15):

$$\varphi = d\mathbf{x}^{123} + d\mathbf{x}^{145} + d\mathbf{x}^{167} + d\mathbf{x}^{246} - d\mathbf{x}^{257} - d\mathbf{x}^{347} - d\mathbf{x}^{356} , \tag{5.1.3}$$

where $d\mathbf{x}^{A_1\ldots A_p} := dx^{A_1} \wedge \ldots \wedge dx^{A_p}$. G_2 also preserves the Hodge dual of this 3-form, $\Phi := *\varphi$. We may thus write

$$\begin{aligned} G_2 &:= \left\{ P \in SO(7) \,\middle|\, P^{A'}_{A} P^{B'}_{B} P^{C'}_{C} \varphi_{A'B'C'} = \varphi_{ABC} \right\} , \qquad (5.1.4) \\ \Rightarrow \mathscr{L}(G_2) &= \left\{ T \in \mathscr{L}(SO(7)) \,\middle|\, T^{A'}_{A}\varphi_{A'BC} + T^{B'}_{B}\varphi_{AB'C} T^{C'}_{C}\varphi_{ABC'} = 0 \right\} . \end{aligned}$$

We can then split the $SO(7)$ generators into a group of 7 not in $\mathscr{L}(G_2)$ and a group of 14 in $\mathscr{L}(G_2)$, which are given by

$$\begin{aligned} F_A &= \frac{1}{2}\varphi_A{}^{BC} S_{BC} , \\ R_{AB} &= \frac{2}{3} S_{AB} - \frac{1}{6}\Phi_{AB}{}^{CD} S_{CD} , \end{aligned} \tag{5.1.5}$$

respectively. This represents the branching rule

$$\mathbf{21}_{SO(7)} \to (\mathbf{14} + \mathbf{7})_{G_2} . \tag{5.1.6}$$

5.1.2 G_2 Projectors

In solving equations containing G_2 spinors, vectors, and tensors, it is usually easiest to project out irreducible representations of the relevant indices. The presence of a globally defined invariant three-form, together with its dual, project out various representations from objects on a G_2 space. Here we summarize our conventions for the projectors that we use.

For a 2-form, ξ, decomposing as $(\mathbf{7}\times\mathbf{7})_{\mathrm{anti-symm.}} \to \mathbf{7}+\mathbf{14}$, we have

$$P_7\xi = \frac{1}{6}(\xi\lrcorner\varphi)\varphi\ , \tag{5.1.7}$$

$$P_{14} = \mathbb{1} - P_7\ . \tag{5.1.8}$$

For a 3-form, ζ, decomposing as $(\mathbf{7}\times\mathbf{7}\times\mathbf{7})_{\mathrm{anti-symm.}} \to \mathbf{1}+\mathbf{7}+\mathbf{27}$ we have

$$P_1\zeta = \frac{1}{42}(\varphi\lrcorner\zeta)\varphi\ , \tag{5.1.9}$$

$$P_7\zeta = -\frac{1}{24}(\zeta\lrcorner\Phi)\lrcorner\Phi\ , \tag{5.1.10}$$

$$P_{27} = \mathbb{1} - P_1 - P_7\ . \tag{5.1.11}$$

For a 4-form, χ, decomposing as $(\mathbf{7}\times\mathbf{7}\times\mathbf{7}\times\mathbf{7})_{\mathrm{anti-symm.}} \to \mathbf{1}+\mathbf{7}+\mathbf{27}$ we have

$$P_1\chi = \frac{1}{168}(\Phi\lrcorner\chi)\Phi\ , \tag{5.1.12}$$

$$P_7\chi = \frac{2}{21}(\varphi\lrcorner\chi)\wedge\varphi\ , \tag{5.1.13}$$

$$P_{27} = \mathbb{1} - P_1 - P_7\ . \tag{5.1.14}$$

For a symmetric rank 2 tensor, t, decomposing as $(\mathbf{7}\times\mathbf{7})_{\mathrm{symmetric}} \to \mathbf{1}+\mathbf{27}$, we have

$$(P_1 t)_{AB} = \frac{1}{7}t^C{}_C g_{AB}\ , \tag{5.1.15}$$

$$P_{27} = \mathbb{1} - P_1\ . \tag{5.1.16}$$

For an $SO(7)$ spinor, decomposing under G_2 as $\mathbf{8}_{SO(7)} \to (\mathbf{1}+\mathbf{7})_{G_2}$ we have

$$P_1 = \frac{1}{8}\left(\mathbb{1} - \frac{1}{4!}\Phi_{ABCD}\gamma^{ABCD}\right)\ , \tag{5.1.17}$$

$$P_7 = \mathbb{1} - P_1\ . \tag{5.1.18}$$

Note that for each of the irreducible representations, there will be a G_2-invariant way of relating the objects in that representation with different indices. So for example a G_2 vector v in the **7** gives a three form $v \lrcorner \Phi$ in the **7**, and so on.

5.1.3 G_2 spinors

In this section, we construct a basis set for general spinors on a G_2 manifold using some of the results from earlier appendices. We also consider the action of our Dirac matrices on these spinors, which is crucial in solving the Killing spinor equations and indeed performing any fermionic calculation.

The 7-dimensional Dirac matrices can be used to provide a representation of $\mathscr{L}(SO(7))$ via

$$\Sigma_{AB} := \frac{1}{2}\gamma_{AB} \ , \tag{5.1.19}$$

which obey the same commutation relations as the S_{AB} above. We can use this isomorphism to split the Σ_{AB} as for section 5.1.1

$$f_A = \frac{1}{2}\varphi_A{}^{BC}\Sigma_{BC} \ , \tag{5.1.20}$$

$$\rho_{AB} = \frac{2}{3}\Sigma_{AB} - \frac{1}{6}\Phi_{AB}{}^{CD}\Sigma_{CD} \ . \tag{5.1.21}$$

We are then able to define a unique spinor η by

$$P_1\eta = \eta \ , \quad P_7\eta = 0 \ , \quad \overline{\eta}\eta = 1 \ . \tag{5.1.22}$$

η is then covariantly conserved on the G_2 manifold, i.e. $\rho_{AB}\chi_0 = 0$. We further define a set of seven other spinors $\{\eta_A\}$ obeying

$$\begin{array}{rclrcl} \eta_A & := & \frac{2}{3}f_A\eta \ , & P_7\eta_A & = & \eta_A \ , \\ \overline{\eta}\eta_A & = & \overline{\eta_A}\eta = 0 \ , & \overline{\eta_A}\eta_B & = & \delta_{AB} \ , \end{array} \tag{5.1.23}$$

which represents the branching rule

$$\mathbf{8}_{SO(7)} \to (\mathbf{7}+\mathbf{1})_{G_2} \ . \tag{5.1.24}$$

We therefore have $\{\eta, \eta_A\}$ as a basis for spinors on a G_2 manifold. The action of our Dirac matrices on these spinors is then given by

$$
\begin{aligned}
\gamma^A\eta &= ig^{AB}\eta_B \ , \\
\gamma^{AB}\eta &= \varphi^{ABC}\eta_C \ , \\
\gamma^{ABC}\eta &= -i\varphi^{ABC}\eta + i\Phi^{ABCD}\eta_D \ , \\
\gamma^{ABCD}\eta &= -\Phi^{ABCD}\eta - 4\varphi^{[ABC}g^{D]E}\eta_E \ , \\
\gamma^{ABCDE}\eta &= -i\left(\Phi^{[ABCD}g^{E]F} + 4\varphi^{[ABC}\varphi^{DE]F}\right)\eta_F \ , \\
\gamma^{ABCDEF}\eta &= -\left(\Phi^{[ABCD}\varphi^{EF]G} + 4\varphi^{[ABC}\Phi^{DEF]G}\right)\eta_G \ , \\
\gamma^{ABCDEFG}\eta &= i\epsilon^{ABCDEFG}\eta - i\left(\Phi^{[ABCD}\Phi^{EFG]H} + 4\varphi^{[ABC}\Phi^{DEF}{}_I\varphi^{G]HI}\right)\eta_H \ .
\end{aligned} \tag{5.1.25}
$$

5.1.4 G_2 identities

By using the above, together with (5.1.22), (5.1.23), Fierz identities and Dirac matrix identities such as those in [124] one can derive some useful formulae relating the forms φ and Φ

$$\varphi_{ABE}\varphi^{ECD} = \Phi_{AB}{}^{CD} + 2\delta^{CD}_{AB} \ , \tag{5.1.26}$$

$$\varphi^{ABF}\Phi_{FCDE} = 6\delta^{[A}_{[C}\varphi^{B]}{}_{DE]} \ , \tag{5.1.27}$$

$$\Phi_{ABCG}\Phi^{DEFG} = 6\delta^{ABC}_{DEF} + 9\Phi_{[AB}{}^{[DE}\delta^{F]}_{C]} - \varphi_{ABC}\varphi^{DEF} \ , \tag{5.1.28}$$

$$36\delta^{[EF}_{[AB}\varphi_{CD]}{}^{G]} = \Phi_{ABCD}\varphi^{EFG} + 4\varphi_{[ABC}\Phi_{D]}{}^{EFG} - \epsilon_{ABCD}{}^{EFG} \ , \tag{5.1.29}$$

$$24\varphi^{[GH}{}_{(A}\delta^{I]}_{B)} = \epsilon^{CDEFGHI}\varphi_{ACD}\varphi_{BEF} \ , \tag{5.1.30}$$

$$\Phi^{CDEF}g_{AB} = 3\varphi_{(A}{}^{[CD}\varphi^{EF]}{}_{B)} + 4\Phi^{[CDE}{}_{(A}\delta^{F]}_{B)} \ , \tag{5.1.31}$$

$$\epsilon_{ABCDEFG} = 5\varphi_{[ABC}\Phi_{DEFG]} \ , \tag{5.1.32}$$

where $\delta^{B_1\ldots B_a}_{A_1\ldots A_a} := g_{[A_1}{}^{[B_1}\cdots g_{A_a]}{}^{B_a]}$. Further identities may be derived by contracting indices in the above. For a 4-form α such that $P_7\alpha = 0$, we have the further result that

$$\alpha \lrcorner \Phi = 4(*\alpha)\lrcorner\varphi \ , \tag{5.1.33}$$

$$(P_{27}\alpha)_{(A}{}^{CDE}\Phi_{B)CDE} = -3(P_{27} * \alpha)_{(A}{}^{CD}\varphi_{B)CD} \ . \tag{5.1.34}$$

Finally, note that one can write the volume of a manifold M_7 with G_2 structure as

$$\mathcal{V} = \frac{1}{7}\int_{M_7} \varphi \wedge \Phi \ . \tag{5.1.35}$$

5.2 Effective action for M-theory compactifications on manifolds with G_2 structure

Having described some features of M-theory compactifications and of G_2 we can go and study M-theory compactifications on manifolds with G_2 structure in more detail. In particular, we want to be able to construct the full effective theory for the $\mathcal{N} = 1$ four-dimensional effective theory that should emerge in the limit of a small compact space.

In this section we shall perform a general analysis that is valid for any manifold with G_2 structure, and obtain information about the Kähler potential and superpotential that appear in such compactifications. These will determine the form of the scalar part of the supergravity (see (4.1.4)). Although the method we use could be applied to other fermionic terms to yield the gauge kinetic function and other quantities relevant to the four-dimensional supergravity, here we restrict ourselves to the scalar sector.

5.2.1 Gravitino mass term

In compactifying string and M-theory down to four dimensions it is usually easier to derive the bosonic terms in the lower-dimensional action. Computing fermionic terms is often more involved and the fermionic part of the action is in general inferred from supersymmetry. There is, however, a specific class of terms that are relatively easy to compute and which give valuable information about the low energy effective action. These terms are the gravitino mass terms, and in $\mathcal{N} = 1$ supergravity in four dimensions they take the form

$$M_{3/2} = \frac{1}{2}e^{\mathcal{K}/2}\Big(\overline{W}\,\overline{\psi^c}_\mu\gamma^{\mu\nu}\psi_\nu + W\overline{\psi}_\mu\gamma^{\mu\nu}\psi^c_\nu\Big) \ , \tag{5.2.1}$$

where the superscript ^{c} stands for Majorana conjugation. $\mathcal{K}$ denotes the Kähler potential and W is the superpotential. By computing such gravitino mass terms one can, therefore, obtain information about the Kähler potential and the superpotential. Similar calculations were performed in [23, 44, 125]. The same information can be obtained by reducing the supersymmetry transformations as was done for example in [126, 127].

Before we start the calculation, some comments about the gravitino are in order. Firstly, its reduction follows the discussion of section 4.2.4, with the spinor η on the compact space M_7 being the only one to produce a low-energy gravitino in the limit of a small compact space. This gives the following ansatz for the gravitino:

$$\Psi_I \propto (\psi_I + \psi_I^c) \otimes \eta \; . \tag{5.2.2}$$

Note that η is a Majorana spinor and so is its own conjugate. The four-dimensional gravitino ψ_μ is taken to be a Weyl spinor, representing the general result that any Majorana spinor χ can be expressed in terms of a Weyl spinor ψ by writing $\chi = \psi + \psi^c$. Taking ψ_μ as positive chirality gives ψ_μ^c as negative chirality.

Secondly, upon reduction there will be a Weyl rescaling $g_{\mu\nu} \to \mathcal{V}^{-1} g_{\mu\nu}$. To keep canonical kinetic terms for the gravitino, the constant of proportionality in (5.2.2) has to be fixed to give

$$\Psi_I = \mathcal{V}^{-1/4} \, (\psi_I + \psi_I^c) \otimes \eta \; . \tag{5.2.3}$$

Finally, note that the relation (5.2.3) yields both the four-dimensional gravitino ψ_μ and also spin-$\frac{1}{2}$ fields ψ_A. Clearly, from the kinetic term in eleven dimensions one obtains a kinetic term for the gravitino, one for the spin-$\frac{1}{2}$ fields and one mixed kinetic term between the gravitino and the spin-$\frac{1}{2}$ fields. In the standard four-dimensional supergravity, such mixed terms are not present and to obtain the correctly normalised fermionic fields one has to perform a further redefinition of the gravitino field which in terms of the eleven dimensional field Ψ_I takes the form

$$\Psi_\mu \to \Psi'_\mu = \Psi_\mu + \delta\Psi_\mu \; , \quad \delta\Psi_\mu \sim \Gamma_\mu \Gamma^A \Psi_A \; . \tag{5.2.4}$$

We note that, since ψ_μ in the above relation appears linearly, the gravitino mass term (5.2.1) for ψ'_μ—when written in terms of the uncorrected gravitino field ψ_μ—has the same form up to terms which are linear in the field ψ_μ. Thus, computing the terms $\psi_\mu\gamma^{\mu\nu}\psi_\nu$, one can still deduce the combination $e^{\mathcal{K}/2}W$ and so in order to make the calculation clearer we shall not be concerned with using the correct definition for the gravitino field (5.2.4).

We can now start to analyse the terms that contribute to the gravitino mass term in four dimensions. We shall have in mind the most general background compatible with Poincaré invariance in four dimensions, which includes internal fluxes $(\hat{F}_4)_{ABCD}$ and a flux in the four space-time dimensions $(\hat{F}_4)_{\mu\nu\rho\sigma}$. Note that the background values for the fermionic fields are taken to vanish and so the four-fermion terms in the eleven-dimensional action cannot contribute to the gravitino mass term in four dimensions. Therefore, we only need to consider the terms that we have kept in the action (3.1.1), which are bilinear in the gravitino field.

In the following we shall analyse one by one these contributions to the gravitino mass term.

Contribution from the kinetic term

As also remarked in [44] the kinetic term for the gravitino only produces a contribution to the mass term in the presence of non-trivial structures, which will be proportional to $D_A\eta$. Inserting the decomposition (5.2.3) in the gravitino kinetic term from (3.1.1) and keeping only the terms which are relevant for the gravitino mass term one finds

$$\begin{aligned}\overline{\Psi}_I\Gamma^{IJK}D_J\Psi_K &= \overline{\Psi}_\mu\Gamma^{\mu A\nu}D_A\Psi_\nu + \text{ terms not contributing to gravitino mass} \\ &= -\mathcal{V}^{-1/2}(\overline{\psi}_\mu+\overline{\psi^c}_\mu)\gamma^{\mu\nu}\gamma(\psi_\nu+\psi^c_\nu)\overline{\eta}\gamma^A D_A\eta+\ldots \end{aligned} \tag{5.2.5}$$

We compute the covariant derivative on the spinor η by making use of (2.2.9) and we find that

$$\overline{\eta}\gamma^A D_A\eta = -\frac{i}{4}\varphi^{ABC}\kappa_{ABC}\ , \tag{5.2.6}$$

where κ_{ABC} denotes the intrinsic contorsion and the G_2 three-form is φ. Note that this quantity picks up the G_2 singlet of the intrinsic contorsion and can be written using the formulæ in chapter 2 as

$$\varphi^{ABC}\kappa_{ABC} \star \mathbf{1} = \frac{1}{2} d\varphi \wedge \varphi \, . \tag{5.2.7}$$

Taking into account the Weyl rescaling, the contribution to the gravitino mass coming from the kinetic term of the gravitino in eleven dimensions can be written

$$M_{3/2}^{(\mathrm{k.t.})} = \left(\frac{i}{16\mathcal{V}^{3/2}} \int d\varphi \wedge \varphi \right) \overline{\psi}_\mu \gamma^{\mu\nu} \psi_\nu^c + \mathrm{c.c.} \tag{5.2.8}$$

Contribution from the internal flux

The next contribution to the gravitino mass term we discuss is the one that appears due to the internal fluxes. Before doing this, we should consider the decomposition of the eleven-dimensional field $\hat{A}_3$ into components. We write this as

$$\hat{A}_3 = \mathring{A}_3 + A_3 + a_3 \, , \tag{5.2.9}$$

where $\mathring{A}_3$ is the background value for $\hat{A}_3$, which is a form on M_7 that depends only on the internal coordinates and gives rise to the background flux, $G := d\mathring{A}_3$, while A_3 and a_3 represent fluctuations around it. A_3 is a purely four-dimensional three-form field and a_3 is a three-form which lives in the internal manifold, but which depends on the external spacetime coordinates as well. From the point of view of the low energy action this form, a_3, will produce scalar fields in four dimensions. Note that in general one can consider also fluctuations that from the four-dimensional perspective are two-forms or vector fields, but as these other fields play no role in the following discussion we ignore them completely. It is important to note, however, that apart from these other degrees of freedom, (5.2.9) is the most general expression one can consider.

The decomposition in (5.2.9) therefore leads to two kinds of fluxes, G and da_3, which we shall absorb into the quantity $F_4 := G + da_3$. The term relevant to the purely internal flux can be written

$$\begin{aligned}\overline{\Psi}_I\Gamma^{IJK_1K_2K_3K_4}\Psi_J(\hat{F}_4)_{K_1K_2K_3K_4} &= \overline{\Psi}_\mu\Gamma^{\mu\nu ABCD}\Psi_\nu(\hat{F}_4)_{ABCD}+\ldots \\ &= \mathcal{V}^{-1/2}(\overline{\psi}_\mu+\overline{\psi^c}_\mu)\gamma^{\mu\nu}(\psi_\nu+\psi^c_\nu)\overline{\eta}\gamma^{ABCD}\eta(F_4)_{ABCD}+\ldots \qquad (5.2.10)\end{aligned}$$

Using (2.2.17) to eliminate the $\overline{\eta}\gamma^{ABCD}\eta$, and taking into account all the factors in the action and the Weyl rescaling, the final result is

$$M^{(\mathrm{i.f.})}_{3/2} = \left(\frac{1}{8\mathcal{V}^{3/2}}\int F_4\wedge\varphi\right)\overline{\psi}_\mu\gamma^{\mu\nu}\psi^c_\nu + \mathrm{c.c.} \qquad (5.2.11)$$

Flux along the external directions

Having a flux for $\hat{F}_4$ completely in the four space-time directions can also generate a mass term for the gravitino from the term in (3.1.1) proportional to

$$\overline{\Psi}^I\Gamma^{JK}\Psi^L(\hat{F}_4)_{IJKL}\ . \qquad (5.2.12)$$

Where we parameterise the purely external flux as

$$(\hat{F}_4)_{\mu\nu\rho\sigma} = f\varepsilon_{\mu\nu\rho\sigma}\ , \qquad (5.2.13)$$

then after all rescalings are taken into account, the contribution to the gravitino mass is simply given by

$$M^{(\mathrm{e.f.})}_{3/2} = -\frac{i}{8}\mathcal{V}^{3/2}f\overline{\psi}_\mu\gamma^{\mu\nu}\psi^c_\nu + \mathrm{c.c.} \qquad (5.2.14)$$

This form for the contribution is, however, slightly misleading. The reason is that we have to dualise the three-form A_3 in a consistent way. A three-form in four dimensions is not dynamical and its dual is thus only an arbitrary constant as discussed in [128] and chapter 8. It is important to stress that even if one chooses this constant to vanish the dualisation of A_3 produces in general a non-trivial result due to its couplings in the four-dimensional action. Thus, in order to obtain the correct result we have to derive first the complete action for the four-dimensional field A_3.

Inserting (5.2.9) into (3.1.1) and also considering the Weyl rescaling we can derive the terms in the low energy action which contain A_3. One is the kinetic term, and it is easy to see that this has the form

$$\mathcal{L}^{\rm kin}_{A_3} = \frac{\mathcal{V}^3}{4} dA_3 \wedge \star dA_3 \ . \tag{5.2.15}$$

The second contribution comes from the Chern-Simons term. Integrating this term over the compact space gives

$$\begin{aligned} \mathcal{L}^{\rm c.s.}_{A_3} &= \frac{1}{12}\int_{M_7} \hat{F}_4 \wedge \hat{F}_4 \wedge \hat{A}_3 \\ &= \frac{1}{2}\int_{M_7}\left(G\wedge a_3 + \frac{1}{2}d_7 a_3 \wedge a_3 + \frac{1}{2} G \wedge \mathring{A}\right) d_4 A_3 =: C d_4 A_3 \ . \end{aligned} \tag{5.2.16}$$

Performing a dualisation as in [128] (and also appendix E.2 of [6]) for the terms (5.2.15) and (5.2.16) gives the condition

$$\frac{1}{2}\mathcal{V}^3 \star d_4 A_3 = \frac{1}{2}\mathcal{V}^3 f = \tilde{\lambda} + C \ , \tag{5.2.17}$$

where $\tilde{\lambda}$ is a genuine constant and C is defined in (5.2.16). We now define one further constant

$$\lambda := 2\tilde{\lambda} + \int_{M_7} G \wedge \mathring{A} \ , \tag{5.2.18}$$

since the integral above is just a constant from the point of view of the four-dimensional theory. Substituting (5.2.18) and (5.2.17) into (5.2.14) then gives

$$M^{\rm (e.f.)}_{3/2} = -\frac{i}{8\mathcal{V}^{3/2}}\left[\lambda + \int_{M_7}\left(G\wedge a_3 + \frac{1}{2} d_7 a_3 \wedge a_3\right)\right]\overline{\psi}_\mu \gamma^{\mu\nu}\psi^c_\nu + \text{c.c.} \tag{5.2.19}$$

5.2.2 The superpotential

We are now in position to compute the superpotential which appears in M-theory compactifications on seven-dimensional manifolds with G_2 structure. Putting together the contributions to the gravitino mass term from (5.2.8), (5.2.11) and (5.2.19), noting that the flux F_4 in (5.2.11) is in fact $G + d_7 a_3$, one finds

$$M_{3/2} = -\frac{i}{16\mathcal{V}^{3/2}}\overline{\psi}_\mu\gamma^{\mu\nu}\psi^c_\nu\left[\int (da_3 + id\varphi)\wedge(a_3 + i\varphi) + 2\lambda + 2\int G\wedge (a_3 + i\varphi)\right] + \text{c.c.} \tag{5.2.20}$$

Using (5.2.1) one immediately obtains (up to an overall phase which plays no role in the definition of the superpotential)

$$e^{\mathcal{K}/2}W = \frac{1}{8\mathcal{V}^{3/2}}\left[\int (da_3 + id\varphi) \wedge (a_3 + i\varphi) + 2\lambda + 2\int G \wedge (a_3 + i\varphi)\right] + \text{c.c.} \tag{5.2.21}$$

This is the first main result of this chapter so let us pause to discuss it for a while. First of all we note that in $\mathcal{N} = 1$ supergravity, the Kähler potential and the superpotential are not truly independent functions, but the only thing which has a physical meaning is the combination $e^{\mathcal{K}}|W|^2$. Obviously the equation above gives us this quantity. However we can not resist noting that this formula looks very suggestive and that it splits up in a natural way in a part which is holomorphic (the part in the brackets) and an overall real factor. It is thus tempting to argue that for general compactifications on manifolds with G_2 structure the Kähler potential for the low energy fields is always given by the same formula as in the case of G_2 holonomy

$$\mathcal{K} = -3\ln\mathcal{V}\ . \tag{5.2.22}$$

Establishing this, the superpotential is then given by

$$W = \frac{1}{8}\left[\int (d\varphi + ida_3) \wedge (\varphi + ia_3) + 2\lambda + 2\int G \wedge (\varphi + ia_3)\right]\ . \tag{5.2.23}$$

One immediately observes in this superpotential that the last term is precisely the result obtained in [15] for the case of manifolds with G_2 holonomy. The second term, which is just a constant, should always be present and appears from the correct dualisation of the field A_3 in four dimensions. It also appears in [15], but in that case only as a quantum effect. Here we only limit ourselves to the supergravity approximation and thus we keep this additional parameter real. Finally one notices the first term, which appears entirely due to the non-trivial G_2 structure (intrinsic torsion). This term is completely new, and in the next section we will compute it explicitly for the case of manifolds with weak G_2 holonomy and also show that it reproduces the potential which can be derived from the dimensional reduction when inserted in the usual $\mathcal{N} = 1$ formula.

Before we end this section we make one more comment on the way this superpotential was derived. In the original action for M-theory (3.1.1), the fermion bilinears coupled only linearly to the field strength $\hat{F}_4$. Naïvely one would conclude from this that the superpotential can depend only linearly on the flux G or the fields which appear in the low energy spectrum from $\hat{A}_3$. However it is clear that the superpotential (5.2.23) contains also quadratic terms like $G \wedge a_3$ or $da_3 \wedge a_3$. Tracing back these terms we see that they only appear from the correct dualisation of the three form A_3 in four dimensions (5.2.19). The observation we want to make is that there is a simpler and more direct way to see such terms appearing by considering the starting action to be the 'duality symmetric' M-theory action [105]. In such a formulation of M-theory one also has a six-form field $\hat{A}_6$ with a seven-form field strength that is defined like

$$\hat{F}_7 = d\hat{A}_6 + \hat{A}_3 \wedge \hat{F}_4 \; . \tag{5.2.24}$$

The fermionic action will now contain fermion bilinears which also couple to $\hat{F}_7$ and from such couplings and also the second term in the definition of $\hat{F}_7$ one can immediately see quadratic terms in $\hat{A}_3$ appearing in the superpotential.

Chapter 6

M-theory on weak G_2 manifolds

Having considered general properties of manifolds with G_2 structure we now analyse the main features of the special case of manifolds with weak G_2 holonomy in the following. We will prove some general results about weak G_2 spaces, in particular about their deformation spaces, before considering the relevance of this result for low-energy physics.

6.1 Weak G_2

Usually not much can be said about generic manifolds with G_2 structure. The characterisation of such manifolds needs in general the introduction of other forms besides φ and Φ in order to parameterise the torsion (see, for example, [73]). However there is a simple case where such manipulations are not necessary. This is the case of manifolds with weak G_2 holonomy, which holds when only the torsion class $\mathcal{W}_1$ of (2.2.20) is nonvanishing. We write this as

$$\begin{aligned} d\varphi &= \tau\Phi \ , \\ d\Phi &= 0 \ . \end{aligned} \tag{6.1.1}$$

The quantity τ is a constant on the manifold with G_2 structure and completely specifies the torsion of a weak G_2 manifold. In the context of M-theory compactifications, as we shall see later, the parameter τ is not completely constant, but it can depend on the space-time coordinates through the volume of the internal manifold.

Since the contorsion is a singlet under the G_2 structure group, it can be written

$$\kappa_{ABC} = \kappa \varphi_{ABC} \ , \tag{6.1.2}$$

where κ is a constant on the manifold with G_2 structure. Note that this is the only possibility we have as φ_{ABC} is the unique three-index object that is a singlet under G_2. It is clear that the contorsion is totally antisymmetric and thus it also coincides with the torsion tensor $T_{ABC} = \kappa_{[AB]C}$. Using (6.1.2) and the fact that φ is covariantly constant with respect to the connection with torsion we can compute the exterior derivative of φ

$$(d\varphi)_{ABCD} = 4\nabla_{[A}\varphi_{BCD]} = 12\kappa_{[AB}{}^{E}\varphi_{CD]E} = 12\kappa\Phi_{ABCD} \ . \tag{6.1.3}$$

We thus see, as anticipated, that the exterior derivative of φ is indeed again a singlet, namely Φ. It will be more convenient to introduce another parameter $\tau = 12\kappa$, obeying

$$\kappa_{ABC} = \frac{\tau}{12}\varphi_{ABC} \ , \tag{6.1.4}$$

in terms of which the weak G_2 condition takes the form (6.1.1).

6.1.1 Weak G_2 identities

Starting from (2.2.19) and using the relations (5.1.27) one can show that

$$\nabla_A\varphi_{BCD} = 3\kappa_{A[B}{}^{E}\varphi_{CD]E} = \frac{\tau}{4}\Phi_{ABCD} \ . \tag{6.1.5}$$

Repeating the procedure for Φ one finds

$$\nabla_A\Phi_{BCDE} = -4\kappa_{A[B}{}^{F}\Phi_{CDE]F} = -\frac{\tau}{3}\phi_{A[B}{}^{F}\Phi_{CDE]F} = -\tau g_{A[B}\varphi_{CDE]} \ . \tag{6.1.6}$$

From the above relation one immediately derives

$$\Box\varphi = \nabla_A\nabla^A\varphi = -\frac{\tau^2}{4}\varphi \ . \tag{6.1.7}$$

In the following we will also need a couple of relations involving the curvature tensors of weak G_2 manifolds. To compute these we start from the fact that the globally defined spinor is covariantly constant with respect to the connection with torsion (2.2.18). Since we work with imaginary gamma matrices the spinor $\gamma^{AB}\eta$ is still a Majorana spinor and thus can be expanded in terms of a basis for Majorana spinors on the weak G_2 manifold $\{\eta, \eta_A\}$ as defined in section 5.1. It is then straightforward to derive that

$$D_A\eta = \frac{1}{4}\kappa_{ABC}\gamma^{BC}\eta = \frac{\tau}{8}\eta_A \ . \tag{6.1.8}$$

Taking the commutator of two covariant derivatives acting on the spinor η one obtains

$$R_{ABCD}\gamma^{CD}\eta = \frac{\tau^2}{8}\gamma_{AB}\eta \ . \tag{6.1.9}$$

Multiplying this from the left with γ^B and using the first Bianchi identity for the Riemann tensor immediately gives the Ricci curvature of weak G_2 manifolds as

$$R_{AB} \equiv R^C{}_{ACB} = \frac{3}{8}\tau^2 g_{AB} \ , \tag{6.1.10}$$

which shows that these manifolds are Einstein. Note, as a matter of fact, that the scaling behaviour of τ found in (6.2.8) is in agreement with the above relation.

Multiplying again (6.1.9) by γ^E and contracting with $\overline{\eta}$ one obtains

$$R_{ABCD}\varphi^{CDE} = \frac{\tau^2}{8}\varphi_{AB}{}^E \ , \tag{6.1.11}$$

which can be used to prove other identities about the Riemann tensor in weak G_2.

6.1.2 Projectors and differential operators

There are some more properties of weak G_2 that will be important for the analysis presented in the paper that we will now prove. We will mainly be interested in three-forms obeying certain conditions whose significance will come to light later.

Before we start there is a useful remark we should make about three-forms on seven-dimensional manifolds (see, for example, [48]): the eigenfunctions of the

Laplace operator corresponding to a non-zero eigenvalue μ^2 are in one-to-one correspondence with the eigenfunctions of the operator $Q := \star d$ corresponding to the eigenvalues $\pm\mu$. The operator Q will play an important role in our analysis as the forms that are relevant for the deformations of weak G_2 manifolds are eigenfunctions of this operator corresponding to the eigenvalue $-\tau$. It is these forms we are going to analyse in the following.

Furthermore the forms we are interested in also do not contain a part which transforms under the **7** of G_2. It would be interesting if one could prove the existence of such forms, but this goes beyond the scope of this work and so for our purposes we will just assume that the forms with these desired properties do indeed exist and prove a result about them.

Let us consider forms Π_i that satisfy

$$\begin{aligned} &\Delta\Pi_i = \tau^2\Pi_i \ , \\ &P_7\Pi_i = 0 \quad \Leftrightarrow \quad \Pi_i \lrcorner \Phi = 0 \ , \end{aligned} \tag{6.1.12}$$

and prove that for such forms the projectors P_1 and P_{27} commute with the differential operator $Q = \star d$. For this, we consider the quantity

$$\Pi_i \lrcorner \varphi \ , \tag{6.1.13}$$

and show that it does not depend on the coordinates of the internal manifold. To see this we compute

$$(d(\varphi \lrcorner \Pi_i))_A = -\frac{\tau}{4}(\Pi_i \lrcorner \Phi)_A - (\varphi \lrcorner d\Pi^i)_A + 3\varphi_{BCD}\nabla^B(\Pi_i)_A{}^{CD} \ . \tag{6.1.14}$$

Using (6.1.12), the first term clearly vanishes. Furthermore, we can choose without loss of generality that the forms Π_i are eigenfunctions of the $Q = \star d$ operator with eigenvalue $\pm\tau$. With this the second term becomes proportional to $\Pi_i \lrcorner \Phi$, which again vanishes for the forms we consider. Thus, we are left with

$$(d(\varphi \lrcorner \Pi_i))_A = 3\varphi_{BCD}\nabla^B(\Pi_i)_A{}^{BC} \ . \tag{6.1.15}$$

The right hand side can be computed if we push φ through the derivative as $d{\star}\varphi = 0$ and then notice that the combination $\varphi_{BCD}(\Pi_i)_A{}^{CD}$ is symmetric in the indices A, B which is again a consequence of (6.1.12). We obtain

$$(d(\varphi\lrcorner\Pi_i))_A = 3\nabla^B(\phi_{BCD}(\Pi_i)_A{}^{CD}) = 3\nabla^B(\phi_{ACD}(\Pi_i)_B{}^{CD}) = 3\frac{\tau}{4}(\Pi_i\lrcorner\Phi)_A\ , \quad (6.1.16)$$

which again vanishes upon using (6.1.12). Note that in the last relation we have also used that $d \star \Pi_i = 0$, which holds true if we take the forms Π to be eigenfunctions of the operator Q. This completes the proof of

$$d(\varphi\lrcorner\Pi_i) = 0\ . \quad (6.1.17)$$

$P_1\Pi_i$ is defined as

$$P_1\Pi_i = \frac{1}{42}(\varphi\lrcorner\Pi_i)\varphi\ , \quad (6.1.18)$$

so we conclude that

$$[P_1, Q]\Pi_i = 0\ . \quad (6.1.19)$$

Since for these forms it also holds that $P_1\Pi_i + P_{27}\Pi_i = \Pi_i$ it follows that

$$[P_{27}, Q]\Pi_i = 0\ . \quad (6.1.20)$$

6.2 Metric deformation space of manifolds with weak G_2 holonomy

So far we have only discussed general features of M-theory compactifications on manifolds with G_2-structure without making any reference to the low-energy field content of such theories. In order to be able to obtain a specific model in four dimensions one needs to have some more information about the internal properties of such manifolds and in particular one needs to get a handle on their moduli space.

In more mathematical literature the word 'moduli' stands for fields which are exactly flat directions of the potential. We shall adopt the more physically motivated use of 'moduli' as fields which appear from gauge-independent metric deformations, subject to some constraint, on the internal manifold. Generically, as we shall see in a while, such fields appear with a potential and it is precisely this feature we are interested in, i.e. we wish to determine the potential for such fields.

In general the question of the moduli space for G_2-structure spaces is quite complicated and a satisfactory answer has not been found yet. For special cases, however, such as the one we will discuss here, it turns out to be possible to gain information about the space of deformations of these manifolds and using this information to compute the low-energy action in four dimensions. It will be our purpose in this section to solve some of the problems outlined here for the case of manifolds with weak G_2 holonomy.

6.2.1 Deformations of the weak G_2 structure

We start by noting that specifying a G_2 structure φ on a seven-dimensional manifold uniquely determines the metric, as in section 2.2.5. Thus, as in the case of G_2 holonomy, the deformations of metrics on manifolds with G_2 structure can be studied by looking at variations of the invariant form φ. For a general G_2 structure the situation is a bit complicated because the torsion itself can vary together with the structure. For this reason, from now on we shall concentrate on the special class of manifolds with weak G_2 holonomy. From (6.1.1) it is clear that the torsion τ cannot depend explicitly on the coordinates of the internal manifold but, from what was said above, it can in principle depend on its moduli and, as we shall see later, in our case it does.

The strategy we adopt is the following. We consider first a G_2 structure φ that satisfies the weak G_2 conditions (6.1.1). Then we consider small variations of this G_2 structure by some arbitrary form $\delta\varphi$ and impose that the equations (6.1.1) are satisfied for some $\tau' = \tau + \delta\tau$. This will yield some conditions on the variations $\delta\varphi$ and $\delta\tau$. It is important to stress here that, by changing φ, the metric on the manifolds changes as well and therefore the action of the Hodge star changes. In order not to create any confusion we will use the notations from [70]. Proposition

10.3.5 from [70] gives the form of the Hodge dual of a perturbed G_2 structure to be

$$\Theta(\varphi + \delta\varphi) = \star\varphi + \frac{4}{3} \star P_1\delta\varphi + \star P_7\delta\varphi - \star P_{27}\delta\varphi \ , \tag{6.2.1}$$

where $\Theta(\xi)$ is a map $\Theta : \Lambda^3 \to \Lambda^4$ which maps to the Hodge dual of a three-form $\xi \in \Lambda^3$ calculated using the metric defined by ξ via (2.2.22). The Hodge star on the right hand side of (6.2.1) is defined from the old (unvaried) metric as are P_1, P_7 and P_{27}, which denote projection operators on the spaces of corresponding dimensionality. Note that on the space of three-forms $P_1 + P_7 + P_{27} = \mathbb{1}$.

Imposing the first relation in (6.1.1) for the varied form $\varphi + \delta\varphi$ at the first order in the perturbations we can write

$$d\delta\varphi = \delta\tau \star \varphi + \tau \left(\frac{4}{3} \star P_1\delta\varphi + \star P_7\delta\varphi - \star P_{27}\delta\varphi \right) \ . \tag{6.2.2}$$

Thus, perturbing a weak G_2 structure φ by some form $\delta\varphi$ leads again to a weak G_2 structure provided the variation $\delta\varphi$ satisfies (6.2.2) for some suitable $\delta\tau$, which for now we assume is an eigenform of the Laplacian, although we will justify this later.

At this stage, we make use of the observation that the **7** component of $\delta\varphi$ makes no contribution to perturbations of the induced metric through the formula (2.2.25). We shall therefore set such components to zero, which will simplify the analysis below. It is shown in section 6.1.2 that for eigenforms of the Laplacian which satisfy $P_7\delta\varphi = 0$, the projectors P_1 and P_{27} commute with the exterior derivative. Since these projectors also commute with the Hodge star one can break (6.2.2) into two simpler conditions for the singlet variations $P_1\delta\varphi$ and for the ones which transform as a **27** under G_2

$$\begin{aligned} dP_1\delta\varphi &= \delta\tau \star \varphi + \frac{4}{3}\tau \star P_1\delta\varphi \ , \\ dP_{27}\delta\varphi &= -\tau \star P_{27}\delta\varphi \ . \end{aligned} \tag{6.2.3}$$

From here it is clear that the torsion τ depends on the deformations of the weak G_2 manifold only via the singlet deformation $P_1\delta\varphi$. In other words, since such singlet deformations only rescale the G_2 structure φ and through it the volume of the manifold, we conclude that the torsion τ depends on the parameters describing the weak G_2 manifold only via its volume.

Let us now consider the two equations above separately. We start with the first one and parameterise the singlet part of the deformation as

$$P_1 \delta\varphi = \epsilon\varphi \ , \tag{6.2.4}$$

for some arbitrary small ϵ. Then the first equation in (6.2.3) becomes

$$\epsilon d\varphi = \delta\tau \star \varphi + \frac{4}{3}\tau\epsilon \star \varphi \ . \tag{6.2.5}$$

Using again the weak G_2 condition (6.1.1) we obtain

$$\delta\tau = -\frac{\epsilon}{3}\tau \ . \tag{6.2.6}$$

From the definition of the volume in terms of the G_2 structure φ (5.1.35) and (6.2.1), it is not hard to see that under the deformation (6.2.4) the volume changes by

$$\delta\mathcal{V} = \frac{7}{3}\epsilon\mathcal{V} \ . \tag{6.2.7}$$

Dividing the last two equations we obtain that the variation of the torsion τ with the volume obeys

$$\frac{\delta\tau}{\tau} = -\frac{1}{7}\frac{\delta\mathcal{V}}{\mathcal{V}} \ , \tag{6.2.8}$$

or after integration

$$\tau \sim \mathcal{V}^{-1/7} \ . \tag{6.2.9}$$

Intuitively the above equation can be understood as follows. As mentioned before, singlet variations rescale the G_2 structure φ. In its turn, such a rescaling produces a rescaling of the metric via (2.2.25) and so the scalar curvature R is rescaled. Since the torsion is directly related to the scalar curvature via (6.1.10) it follows that such a deformation can only be present if the torsion of the manifold itself changes and the quantitative measure of this change is captured by the above equation.

Let us now turn our attention to the second equation in (6.2.3). As we said before, in the case of variations with forms which transform as **27** under G_2 the torsion does not vary. Intuitively, since the torsion is a singlet under G_2 we would need another object which transforms non-trivially under G_2 in order to produce a singlet out of the deformation $\delta\varphi$. Since we do not have at our disposal any such thing it can be understood that the torsion can not change under such deformations. Thus, the second equation in (6.2.3) only imposes a condition on the variations of φ which are compatible with the weak G_2 structure.

So far we have learnt that the non-trivial metric deformations of weak G_2 manifolds are parameterised by three-forms $\delta\varphi$ satisfying

$$\begin{aligned} P_7\delta\varphi &= 0 \ , \\ dP_{27}\delta\varphi &= -\tau \star P_{27}\delta\varphi \ . \end{aligned} \tag{6.2.10}$$

Note the counter-intuitive minus sign appearing on the right hand side which is going to be crucial in determining the correct mass of the modes associated with these variations of the G_2 structure.

Let us now try to give a more explicit parameterisation of the deformation space. Recall that, for the case of manifolds with G_2 holonomy, the form φ was closed and coclosed and thus harmonic. Consequently this form is expanded in a basis for harmonic three-forms with the coefficients being the moduli fields. Then, by general methods, which we present in section 6.2.2, we can compute the metric on the moduli space. Let us try to do something similar here. Clearly from the relations (6.1.1) we see that the form φ cannot be harmonic anymore and in fact the torsion τ measures its failure to be harmonic. However, it is easy to compute the action of the Laplace operator on φ and one obtains

$$\Delta\varphi := (\star d \star d + d \star d\star)\varphi = \tau^2\varphi \ , \tag{6.2.11}$$

and thus φ is still an eigenform of the Laplace operator corresponding to the eigenvalue τ^2. Consider a basis Π_i for the three-forms satisfying

$$\Delta\Pi_i := (\star d \star d + d \star d\star)\Pi_i = \tau^2\Pi_i \ . \tag{6.2.12}$$

There are some concerns about whether these forms depend on the moduli of the weak G_2 manifold that will be dealt with in section 6.2.4. For now, however, we expand the G_2 structure φ in this basis so that

$$\varphi = s^i \Pi_i \,, \tag{6.2.13}$$

and, as for manifolds of G_2 holonomy, we think of the forms Π_i as independent of the choice of metric. For the parameters s^i to be truly moduli of the weak G_2 structure, we still need the forms Π_i to satisfy the condition (6.2.10). From here on we will assume that the forms Π_i used in the expansion (6.2.13) do satisfy this condition as well. If this is true, then the coefficients s^i are indeed the scalar fields which characterise the possible deformations of the weak G_2 manifold.

As one does on a manifold with G_2 holonomy, let us further define

$$k_i = \int \Pi_i \wedge \Phi \,. \tag{6.2.14}$$

Using the fact that the forms Π_i satisfy (6.2.10) one derives

$$\begin{aligned} d\Pi_i &= d(P_1\Pi_i + P_{27}\Pi_i) \\ &= dP_1\Pi_i - \tau \star P_{27}\Pi_i \\ &= -\tau \star \Pi_i + 2dP_1\Pi_i \,. \end{aligned} \tag{6.2.15}$$

The projector on the singlet subspace P_1 is defined by

$$P_1\Pi_i = \frac{1}{7\mathcal{V}} \int \Pi_i \wedge \Phi = \frac{k_i}{7\mathcal{V}} \,, \tag{6.2.16}$$

where we have used that $\Pi_i \lrcorner \varphi$ does not depend on the internal manifold, fact which is also proved in section 6.1.2 above. Using the weak G_2 conditions (6.1.1) one immediately finds

$$d\Pi_i = -\tau \star \Pi_i + \frac{2\tau k_i}{7\mathcal{V}}\Phi \,. \tag{6.2.17}$$

This is the central relation of this section as it will allow us to compute explicitly the low-energy effective action for compactifications on manifolds with weak G_2 holonomy and compare it with the result derived before on general grounds.

It is important to stress one more thing here. At this stage we cannot say much more about the space of deformations of weak G_2 manifolds, apart from the fact that it can be characterised in terms of the forms Π_i satisfying (6.2.10), which will be exploited in the next section. One thing is, however, quite important. Note that the operator $(\Delta - \tau^2)$ is an elliptic operator. It is known in the general theory of operators that elliptic operators on compact manifolds have a finite dimensional kernel. This means that at least the space of deformations of weak G_2 manifolds is finite.

Before we move on let us summarise the main results that we derived in this section. For weak G_2 manifolds the invariant form φ turns out to be an eigenform of the Laplace operator (6.2.11). Performing the expansion in terms of the deformations in the same way as one did on manifolds with G_2 holonomy we conclude that the forms in which we perform this expansion must obey (6.2.10). It is interesting to note that this relation was obtained in a pretty generic fashion. Similar relations were derived for half-flat manifolds [73] by using mirror symmetry as an additional source of information about manifolds with $SU(3)$ structure. Those relations were, however, valid only in a certain limit which was denoted as the small torsion limit while the relations above are valid for any torsion. We will see later on that the torsion cannot be small as it turns out to be of the order of the inverse radius of the manifold.

6.2.2 Metric on the deformation space

Let us suppose that we have expanded φ in terms of some parameters s^i

$$\varphi = s^i \Pi_i \ , \tag{6.2.18}$$

where Π_i are three forms which further satisfy

$$P_7 \Pi_i = 0 \ , \tag{6.2.19}$$

and which do not depend on the parameters s^i. The meaning of the above equation is that, using the set of forms Π_i, one can parameterise the deformations of the G_2-structure φ in terms of variations of the parameters s^i. These parameters, or more correctly their variations, are constant on the internal manifold, but from a four-dimensional perspective they become (scalar) fields. We have seen in the previous section that variations of the G_2 structure induce variations in the metric on the manifold via (2.2.25) and so we can say that the parameters s^i describe the metric fluctuations on the internal manifold. Consequently the kinetic term for these fields in four dimensions appears from the expansion of the eleven-dimensional Ricci scalar. From (4.2.8) and (4.2.7), this gives

$$\begin{aligned}\int_{M_{11}} \sqrt{-g} R_{11} \quad \rightarrow \quad & \int_{M_{11}} \sqrt{-g} \Big\{ R_4 + \mathcal{V}^{-1} R_7 + \frac{3}{2} \partial_\mu (\ln \mathcal{V}) \partial^\mu (\ln \mathcal{V}) \\ & + \frac{1}{4\mathcal{V}} g^{AB} g^{CD} \left(\partial_\mu g_{AB} \partial^\mu g_{CD} - \partial_\mu g_{AC} \partial^\mu g_{BD} \right) \Big\} , \quad (6.2.20)\end{aligned}$$

where the arrow involves performing the Weyl rescaling. Inserting (2.2.25) in the last line of the equation above, we get

$$\int_{M_{11}} \sqrt{-g} \frac{1}{4\mathcal{V}} \left[\left(\frac{1}{2} \varphi_{(A}{}^{CD} \partial_\mu \varphi_{B)CD} - \frac{1}{18} (\varphi \lrcorner \partial_\mu \varphi) g_{AB} \right)^2 - \frac{1}{81} (\varphi \lrcorner \partial_\mu \varphi)(\varphi \lrcorner \partial^\mu \varphi) \right] , \quad (6.2.21)$$

where the $()^2$ represents contraction over both the A, B indices and the μ index. Inserting (6.2.18) into (6.2.21) and using (5.1.28), one obtains

$$S^4_{\text{kin}} = \frac{1}{4\mathcal{V}} \int_{M_7} \Pi_i \wedge \star \Pi_j \int_{M_4} \sqrt{-g}\, \partial_\mu s^i \partial^\mu s^j . \quad (6.2.22)$$

The sigma-model metric for the scalars s^i in four dimensions is then given by

$$g_{ij} = \frac{1}{4\mathcal{V}} \int_{M_7} \Pi_i \wedge \star \Pi_j . \quad (6.2.23)$$

Since the resulting four-dimensional action is supersymmetric, the above metric has to be Kähler. This is indeed the case and the Kähler potential was derived on general grounds in section 5.2.2 is

$$\mathcal{K} = -3 \ln(\mathcal{V}) , \quad (6.2.24)$$

where the volume $\mathcal{V}$ was defined in (5.1.35). To show that this is the Kähler potential corresponding to the metric (6.2.23) we have to know the dependence of the volume on the parameters s^i. From (6.2.18), φ depends linearly on s^i, provided the forms Π_i are independent of these parameters. For the form Φ this dependence is more complicated because of the Hodge duality operation which is involved, but its variation with the parameters s^i can be read off from (6.2.1). One finds

$$\frac{\delta\mathcal{V}}{\delta s^i} = \frac{1}{7}\int_{M_7} \Pi_i \wedge \Phi + \frac{1}{7}\int_{M_7} \varphi \wedge \star\frac{4}{3}\Pi_i = \frac{1}{3}\int_{M_7} \Pi_i \wedge \Phi \ . \tag{6.2.25}$$

With this, one immediately finds the first derivative[1] of the Kähler potential (6.2.24)

$$\begin{aligned} \mathcal{K}_i := \partial_i \mathcal{K} &= \frac{1}{2}\frac{\partial}{\partial s^i}\left(-3\ln(\mathcal{V})\right) \\ &= \frac{-1}{2\mathcal{V}}\int \Pi_i \wedge \Phi \ . \end{aligned}$$

Using again the relation (6.2.1) we can compute the second derivative of the Kähler potential

$$\begin{aligned} \mathcal{K}_{i\bar{j}} &= \frac{1}{4}\left(\frac{\partial\mathcal{V}}{\partial s^j}\mathcal{V}^{-2}\int \Pi_i \wedge \Phi - \mathcal{V}^{-1}\int \Pi_i \wedge \left(\frac{4}{3}\star P_1\Pi_j - \star P_{27}\Pi_j\right)\right) \\ &= \frac{1}{4\mathcal{V}}\int \Pi_i \wedge \star\Pi_j = g_{ij} \ , \end{aligned} \tag{6.2.26}$$

where we have used (6.2.19) and (6.2.25). As anticipated we see now that the metric (6.2.23) can indeed be derived from the Kähler potential (6.2.24). We should stress here that this result is quite general and holds as long as the forms Π_i do not depend on the parameters s^i and satisfy (6.2.19).

6.2.3 Useful formulæ on the deformation space of weak G_2 manifolds

Before we perform the compactification we will find it useful to derive some formulæ which make the calculation on the deformation space of weak G_2 manifolds easier.

[1]Note that a Kähler potential makes sense only in the context of complex geometry. Thus what we have in mind here is that the Kähler potential (6.2.24) is a function of the complex fields T^i defined in (6.3.13) and thus derivatives are then taken with respect to the fields T^i rather then only their imaginary parts s^i.

The presence of the forms Π_i which are not closed (although they are co-closed) allows us to introduce a topological, two-index, symmetric object on these manifolds

$$k_{ij} = \int \Pi_i \wedge d\Pi_j = k_{ji} \ . \tag{6.2.27}$$

Obviously, the appearance of such a matrix is only due to the non-minimal structure as it depends on $d\Pi_i$, which would clearly vanish for the case of manifolds with G_2 holonomy. As we shall see later on this object will enter the expression of the superpotential in terms of the low-energy fields.

A straightforward calculation, which we have outlined in section 6.2.2, shows that for a general expansion of the form (6.2.13), the sigma model metric for the moduli takes the form

$$g_{ij} = \frac{1}{4\mathcal{V}} \int \Pi_i \wedge \star\Pi_j \ . \tag{6.2.28}$$

Using (6.2.17) and (6.2.27) it is easy to show that

$$g_{ij} = -\frac{1}{4\tau\mathcal{V}} k_{ij} + \frac{k_i k_j}{14\mathcal{V}^2} \ . \tag{6.2.29}$$

Furthermore one also has the usual relations

$$\begin{aligned} k_i s^i &= 7\mathcal{V} \ , \\ k_i &= 4\mathcal{V} g_{ij} s^j \ , \\ k_i g^{ij} &= 4\mathcal{V} s^j \ . \end{aligned} \tag{6.2.30}$$

The matrix k_{ij} introduced in (6.2.27) can be shown to satisfy

$$\begin{aligned} k_{ij} s^j &= \tau k_i \ , \\ k_{ij} g^{jk} &= -4\tau\mathcal{V}\delta_i^k + \frac{8}{7}\tau k_i s^k \ . \end{aligned} \tag{6.2.31}$$

Using these relations we can now proceed and compute the effective action which arises by compactifying M-theory on manifolds with weak G_2 holonomy.

6.2.4 Dependence of forms on moduli

There is one more aspect which is crucial in the whole construction so far, namely the dependence on the parameters s^i, introduced in (6.2.13), of the basis of forms we consider $\{\Pi_i\}$. In principle there is no reason to believe that the three-form solutions of the equation $\Delta = \tau^2$ are independent of the parameters s^i of the manifold as the metric itself depends on such forms. In fact when one does a variation of the metric the operator Δ changes and so we expect its eigen-forms to change as well. This also happens for ordinary manifolds with restricted holonomy like Calabi-Yau manifolds or manifolds with G_2 structure. In these cases however, one can easily show that such a dependence on the moduli of the harmonic forms is exact. If one assumes that the same happens for the case of forms that are eigenvalues of the Laplace operator corresponding to non-zero eigenvalues then it is quite easy to show that such a 'mild' dependence on the parameters is not going to affect the results we have derived so far.

First of all it is straightforward to see that this dependence drops out completely from the definition of k_{ij}. The other thing to show is that the metric on the deformation space does not get an additional dependence on the parameters from the forms Π_i. Indeed, if such a dependence on the parameters s^i of these forms is only via an exact form one can immediately see that

$$\int \delta_{(s^k)}\Pi_i \wedge \star\Pi_j = \int d\beta_{i,k} \wedge \star\Pi_j = -\int \beta_{i,k} \wedge d\star\Pi_j = 0 \; , \tag{6.2.32}$$

because the forms Π_i are coclosed. We thus conclude that the only relevant dependence on the parameters of the weak G_2 manifold s^i, is via the expansion (6.2.13) as we considered in the main text.

6.3 M-theory compactifications on manifolds with weak G_2 holonomy

Having discussed in the previous section the possible deformations of weak G_2 manifolds we shall now move on and derive the low-energy action which appears when compactifying M-theory on such manifolds. Then we will show that the resulting theory is an $\mathcal{N} = 1$ supergravity coupled to chiral multiplets. The corresponding

Kähler potential and superpotential will turn out to be the ones derived on general grounds in section 5.2.2.

6.3.1 The compactification

To perform the compactification on manifolds with weak G_2 holonomy one has first to identify the fields which appear in four dimensions. In the previous section we have argued that the AdS-massless scalars which appear in compactifications on manifolds with weak G_2 holonomy are given by the expansion in forms which satisfy (6.2.10). Neglecting, as in the usual Kaluza-Klein setup, the rest of the massive towers of states we can now perform the compactification on weak G_2 manifolds and keep only the modes discussed above.

From the expansion (6.2.13) and the relations in section 2.2.5 one can derive what will be the kinetic term for the scalars s^i which comes from the expansion of the Ricci scalar. As in the case of manifolds with G_2 holonomy the sigma-model metric takes the form

$$g_{ij} = \frac{1}{4\mathcal{V}} \int \Pi_i \wedge \star \Pi_j \; . \tag{6.3.1}$$

In the matter sector we perform a similar expansion to (6.2.13). In this paper we will only be interested in the scalar fields which arise in the compactification of M-theory on a manifold with weak G_2 holonomy. There can be also other fields like vectors, but here we will ignore them completely. If we denote again as in (5.2.9) the internal component of the field A_3 by a_3, then we write

$$a_3 = a^i \Pi_i \; . \tag{6.3.2}$$

The full eleven-dimensional three form $\hat{A}_3$ then takes the form

$$\hat{A}_3 = A_3 + a^i \Pi_i \; , \tag{6.3.3}$$

where A_3 is a three-form in four dimensions. This is not dynamical and so it can be dualised to a constant as we did in section 5.2. The four-dimensional bosonic action derived in this way has the form

$$S_4 = \frac{1}{2}\int \left[\sqrt{-g}R - g_{ij}dT^i \wedge \star d\overline{T}^j - \sqrt{-g}V\right] \ . \tag{6.3.4}$$

The potential V comes from three distinct places. First it comes from the purely internal part of of $\hat{F}_4$. This will have the form

$$V_1 = \frac{1}{8\mathcal{V}^2}\int da_3 \wedge \star da_3 \ , \tag{6.3.5}$$

where the exterior derivative is understood to be in the internal manifold direction and the factor $1/\mathcal{V}^2$ comes from the Weyl rescaling in four dimensions. Using (6.2.17) this can be easily seen to be

$$V_1 = \frac{\tau^2}{8\mathcal{V}^2}a^i a^j \int \Pi_i \wedge \star\Pi_j = \frac{\tau^2}{2\mathcal{V}}a^i a^j g_{ij} \ . \tag{6.3.6}$$

From the dualisation of A_3 in four dimensions we have already seen that there is a contribution to the potential (5.2.19). Using (6.3.2) and (6.2.17) we find[2]

$$V_2 = \frac{1}{4\mathcal{V}^3}\left(\lambda - \frac{a^i a^j k_{ij}}{2}\right)^2 \ . \tag{6.3.7}$$

Finally one has to take into account the contribution from the curvature of the internal manifold. The Ricci scalar of weak G_2 manifolds can be easily computed (6.1.10), and after performing the integration over the internal manifold whilst taking into account the factor $1/\mathcal{V}^2$ coming from the Weyl rescaling in four dimensions one obtains

$$V_3 = -\frac{21\tau^2}{16\mathcal{V}} \ . \tag{6.3.8}$$

The potential coming from the compactification thus takes the form

$$\begin{aligned} V &= V_1 + V_2 + V_3 \\ &= \frac{1}{16}\left[-21\frac{\tau^2}{\mathcal{V}} + \frac{1}{\mathcal{V}^3}(a^i a^j k_{ij})^2 + 16\frac{\tau^2}{\mathcal{V}}a^i a^j g_{ij}\right] \ . \end{aligned} \tag{6.3.9}$$

[2]Note that we now take the dual of A_3 in four dimensions to be the constant λ, which can in principle be independent of the background value for $(\hat{F}_4)_{\mu\nu\rho\sigma}$ as discussed above.

6.3.2 Comparison with the general result

To conclude this analysis we still have to show that the result obtained in the previous subsections is indeed an $\mathcal{N}=1$ supergravity. As we have neglected completely the gauge fields we only have to find the corresponding Kähler potential and superpotential. This is not a hard task since we have at our disposal the general result derived in section 5.2.2. In section 6.2.2 above it was shown that the metric (6.3.1) is Kähler, i.e. $g_{ij} = \partial_i \partial_{\bar{j}} \mathcal{K}$, and the Kähler potential is

$$\mathcal{K} = -3 \ln \mathcal{V} \; . \tag{6.3.10}$$

As we argued in section 5.2.2, the superpotential is given by (5.2.23), which for the case where a non-trivial structure but no G fluxes are taken into account becomes

$$W = \frac{1}{8} \int_{M_7} d(a_3 + i\varphi) \wedge (a_3 + i\varphi) + \frac{\lambda}{2} \; . \tag{6.3.11}$$

Using the field expansions (6.2.13) and (6.3.2) and also the definition (6.2.27) we obtain the superpotential in terms of the four-dimensional fields

$$W = \frac{k_{ij}}{8} T^i T^j + \frac{\lambda}{2} \; , \tag{6.3.12}$$

where the complex fields T^i are defined as

$$T^k = a^k + i s^k \; . \tag{6.3.13}$$

To show that the action (6.3.4) is the bosonic part of an $\mathcal{N}=1$ supergravity theory we have to show that the potential (6.3.9) can be derived from the superpotential (6.3.12) using the general supergravity formula

$$V = e^{\mathcal{K}} \left[D_i W D_{\bar{j}} \overline{W} g^{\bar{j}i} - 3|W|^2 \right] \; , \tag{6.3.14}$$

where as usual D denotes the Kähler covariant derivative.

The calculation is a bit tedious, but completely straightforward and so we will present only the main steps in the following. First one can derive

$$D_i W = \frac{1}{4} k_{ij} T^j + \frac{i}{2} \frac{k_i}{\mathcal{V}} W \; . \tag{6.3.15}$$

Using now the formulæ from section 6.2.3 one easily finds that

$$D_iWD_{\bar{j}}\overline{W}g^{\bar{j}i} = \frac{1}{4}(-\tau\mathcal{V}k_{ij} + \frac{2}{7}\tau^2 k_i k_j)T^i\overline{T}^j - \frac{i}{2}\tau k_i(T^i\overline{W} - \overline{T}^i W) + 7|W|^2\ . \quad (6.3.16)$$

Furthermore, one shows that

$$\frac{i}{2}\tau k_i(T^i\overline{W} - \overline{T}^i W) = -7\tau\mathcal{V}\mathrm{Re}(W) + 4(\mathrm{Im}(W))^2\ . \quad (6.3.17)$$

Finally, one obtains

$$4|W|^2 - \frac{i}{2}\tau k_i(T^i\overline{W} - \overline{T}^i W) = \frac{1}{16}\left[(a^ia^jk_{ij})^2 - (s^is^jk_{ij})^2\right]\ . \quad (6.3.18)$$

Putting all the results together one obtains the final form of the potential

$$V = \frac{1}{16}\left[-21\frac{\tau^2}{\mathcal{V}} + \frac{1}{\mathcal{V}^3}(a^ia^jk_{ij})^2 + 16\frac{\tau^2}{\mathcal{V}}a^ia^jg_{ij}\right]\ , \quad (6.3.19)$$

which is precisely the potential derived in (6.3.9) from the compactification side.

To conclude, we have shown in this section that the compactification of M-theory on a manifolds with weak G_2 holonomy leads to an $\mathcal{N} = 1$ supergravity coupled to chiral superfields in four dimensions with Kähler potential defined by (6.3.10) and superpotential (6.3.12). This is a nice test of the general analysis of M-theory compactifications on manifolds with G_2 structure presented in section 5.2, where the superpotential was derived from computing the four-dimensional gravitino mass term.

6.3.3 Mass operators in AdS

In general Kaluza-Klein compactifications one first identifies the massless modes and truncates away the massive towers of modes which appear. For this it is necessary to identify correctly the mass terms for the various fields in the theory. For compactifications on G_2 manifolds, which are Ricci flat, this is a straightforward exercise and the masses of the different modes can be obtained by studying the spectrum of the Laplace operators acting on various degree forms and of the Lichnerowicz operator.

For the case at hand the situation is a bit more complicated because of the fact that manifolds with weak G_2 holonomy have a non-vanishing Ricci curvature, while the external space is AdS rather than Minkowski. As discussed in section 3.3.2, compactifications on manifolds with weak G_2 holonomy can be understood in terms of the Freund-Rubin solution. The general analysis for the compactification of M-theory in the Freund-Rubin background was performed in [48], and so we will just adapt the formulæ used there for our own purposes.

In our analysis we will only be interested in the scalar fields which appear in the four-dimensional effective action. The relevant mass operators were computed for example in [48] and for the conventions we use in this paper they take the form

$$\begin{aligned} M^2_{0^-} &= Q^2 + \frac{3}{2}\tau Q + \frac{1}{2}\tau^2 = (Q+\tau)(Q+\frac{1}{2}\tau), \\ M^2_{0^+} &= \Delta_L - \frac{1}{4}\tau^2 = Q(Q+\frac{1}{2}\tau) \ , \end{aligned} \tag{6.3.20}$$

where Q stands for the eigenvalue of the operator $\star d$ acting on the internal three-forms in which the scalars are expanded. The first of these operators is for three-form matter fields. The second is for traceless symmetric variations of the metric and involves a tedious calculation using the Lichnerowicz operator Δ_L on such variations.

The presence of supersymmetry should complexify the G_2 structure φ by the matter three-form a_3, and therefore to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions one needs to expand a_3 in the same way as φ. It is not hard to see that the above formulæ for the masses of the scalars coming from the metric deformations and $\hat{A}_3$ matches the mass pattern of a Wess-Zumino (chiral) multiplet in AdS space [129], confirming our expectations about supersymmetry.

Wess-Zumino multiplets are the AdS equivalents of chiral multiplets. From general considerations for AdS_4, $Q < -\frac{1}{4}\tau$, however we conjecture that there will be a more restrictive bound coming from the analysis of weak G_2 spaces. There are already results about the Dirac operator on such spaces, e.g. [130], and the fermionic components of the multiplet have mass operator

$$M^2_{1/2} = (Q+\frac{1}{2}\tau)^2 \ , \tag{6.3.21}$$

where Q is used to label the Wess-Zumino multiplet, since fermions are not expanded in terms of form fields. We have not proved such a conjecture, but hope that it will support the idea that $Q = -\tau$ corresponds to the appropriate low-energy scalar degrees of freedom.

6.3.4 Inclusion of non-vanishing flux

In the previous section, we didn't consider the contributions to the superpotential arising from the internal flux G, since this has already been covered for G_2 holonomy in [131] and we are confident that the term in the superpotential is correct. It is still, however, necessary to consider the forms in which it will be appropriate to expand the flux G. Firstly, note that following the discussion of the previous section, we have expanded the three-form leading to four-dimensional scalars as

$$T = \varphi + ia_3 = T^i \Pi_i \ . \tag{6.3.22}$$

It is then clear that the part of the superpotential arising from G-flux in (5.2.23) takes the form

$$W_{\text{flux}} = \frac{1}{4} \int G \wedge T \propto \langle \star G, T \rangle \ , \tag{6.3.23}$$

where $\langle , \rangle$ denotes the inner product for forms. Clearly, this quantity will vanish unless $\star G$ has the same eigenvalue of $Q = \star d$ as T does. [3]Now, to be consistent we can only consider G at linear level, so that we do not have to take the back-reaction on the geometry into account. In this regime, from [48], we have that

$$Q \star G = -\frac{3}{2} \tau \star G \ . \tag{6.3.24}$$

Our conclusion from this is that the flux decouples from the scalar modes that we have been considering, i.e. those for which $Q = -\tau$, and only couples to the more massive modes with $Q = -\frac{3}{2}\tau$.

[3]Note that the operator Q acting on 3-forms on a seven-dimensional manifold is self-adjoint as $Q^\dagger = (\star d)^\dagger = d^\dagger \star = \star d \star \star = \star d = Q$.

Chapter 7

G_2 domain walls in M-theory

The plan of this chapter is as follows. In Section 7.1 we derive the first-order differential equations describing the G_2 domain walls. In Section 7.2, these equations are solved explicitly in terms of a mode expansion on the G_2 space. The inclusion of membrane and M5-brane sources is discussed in Section 7.3. In Section 7.4, we review the four-dimensional $\mathcal{N} = 1$ supergravities obtained by compactifying M-theory on G_2 spaces with flux, and find their domain wall solutions. These solutions are then compared to their eleven-dimensional counterparts.

It can be shown [132,133] that a solution to the Killing spinor equation $\delta_\epsilon \Psi_I = 0$ which also satisfies the form-field equation of motion (3.1.3) and the Bianchi identity (3.1.2) provides a solution to the Einstein equation (3.1.4) as well. Since Killing spinor equations are typically linear, first-order differential equations, in contrast to the non-linear, second-order equations of motion, our method for finding solutions shall be the use of Killing spinor equations.

7.1 Finding supersymmetric G_2 domain wall solutions

7.1.1 General considerations

In the absence of flux, the general M-theory backgrounds which lead to four-dimensional $\mathcal{N} = 1$ supersymmetry consist of a direct product of a G_2 manifold and four-dimensional Minkowski space. The main goal of this chapter is to understand how these backgrounds are modified in the presence of flux. As is well-known [15], flux leads to a non-vanishing moduli superpotential in the associated four-dimensional effective theory. The "simplest" solution of this theory is then a domain wall [134] rather than four-dimensional Minkowski space. We will return to this four-dimensional viewpoint later. For the eleven-dimensional Ansatz in the presence of flux, this observation suggests we should accordingly modify its four-dimensional Minkowski space part to a domain wall. As we will see, the metric on the G_2 space also requires a correction due to flux.

In practise, we will work with the Killing spinor equations, the equation of motion (3.1.3) for $\hat{F}_4$ and its Bianchi identity (3.1.2). To simplify the problem, the flux is regarded as an expansion parameter and we will determine the flux-induced corrections to linear order. The logic of the calculation is somewhat similar to [87, 90], where flux corrections to Calabi-Yau backgrounds were determined. The main result of this section will be a set of first order bosonic differential equations for these linearised corrections.

7.1.2 Covariantly constant spinors

As noted above, to obtain supersymmetric solutions we impose that the variation of the gravitino (3.1.5) should vanish. In the zero-flux regime, where we just consider the direct product of Minkowski space with a G_2 manifold, $M_4 \times M_7$, this amounts simply to imposing that there should be a covariantly constant spinor, η, obeying

$$D_I\eta = \partial_I\eta + \frac{1}{4}\omega_I{}^{\underline{IJ}}\Gamma_{\underline{IJ}}\eta = 0 \ , \tag{7.1.1}$$

where $\omega_I{}^{\underline{IJ}}$ is the spin connection. In Chapter 2, we explained why G_2 manifolds will in general admit such a spinor. When we introduce flux into the equation (3.1.5), however, the condition on the Killing spinor becomes more complicated. This will lead us to perturb both the metric and spinor in order to preserve supersymmetry.

7.1.3 Metric Ansatz

Following the earlier discussion, we shall consider solutions to M-theory with line element corresponding to a warped product of an internal seven-dimensional space and a domain wall in four dimensions, that is,

$$ds^2 = e^{2\alpha}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2\beta}dy^2 + g_{AB}dx^A dx^B \ . \tag{7.1.2}$$

The three-dimensional part of the metric corresponds to the domain wall worldvolume X_3 spanned by coordinates x^μ, and y is a coordinate transverse to the wall. The seven-dimensional internal space M_7 with coordinates x^A has a metric g_{AB}. Along with the warp factors α and β it generally depends on x^A and y but not on x^μ. Therefore, we have preserved three-dimensional Poincaré invariance on the domain wall worldvolume, a general requirement which we will later use to constrain the flux.

As we have explained, we would like to find a solution with metric of the form above expanding to linear order in the flux. We should, hence, think of the metric (7.1.2) as a linear perturbation of a direct product of Minkowski space with a G_2 space with Ricci-flat metric. To this end, we expand to linear order in the warp factors α and β and write the internal seven-dimensional metric as $g_{AB} = g^{(0)}_{AB} + h_{AB}$, where $g^{(0)}$ is a Ricci-flat metric on a G_2 space and h is the perturbation. The metric (7.1.2) then takes the form

$$ds^2 = (1+2\alpha)\eta_{\mu\nu}dx^\mu dx^\nu + (1+2\beta)dy^2 + (g^{(0)}_{AB} + h_{AB})dx^A dx^B \ . \tag{7.1.3}$$

Note that to zeroth order—that is setting α, β and h to zero, and in the absence of flux—this metric indeed provides a supersymmetric solution to M-theory for the reasons given in chapter 3. When we perturb the metric $g \mapsto g + h$, for linear h, we also perturb the spinor covariant derivative [87] as

$$D^I \mapsto D^I - \frac{1}{8}\left(\nabla_J h^I{}_K - \nabla_K h^I{}_J\right)\Gamma^{JK} \ . \tag{7.1.4}$$

Hence, if we want (3.1.5) to hold in the presence of flux, we should think of the corrections α, β and h as being "sourced" by flux. Our goal will be to determine their explicit form as a function of the flux, such that the corrected solution continues to preserve some supersymmetry.

7.1.4 Conditions on the flux

We will now write down the general form of the flux and the constraints imposed on it by the $\hat{F}_4$ equation of motion and the Bianchi identity.

Given that we are asking for Poincaré invariance on the domain wall worldvolume X_3, we are left with the following non-trivial components of $\hat{F}_4$:

$$\begin{array}{rclcrcl} (\hat{F}_4)_{ABCD} & = & G_{ABCD} \ , & \qquad & (\hat{F}_4)_{yABC} & = & J_{ABC} \ , \\ (\hat{F}_4)_{A\mu\nu\rho} & = & V_A \varepsilon_{\mu\nu\rho} \ , & \qquad & (\hat{F}_4)_{y\mu\nu\rho} & = & K\varepsilon_{\mu\nu\rho} \ . \end{array} \tag{7.1.5}$$

Note that G, J, V, K can be viewed as forms of various degree on the internal space M_7.

Within the context of our expansion scheme, we consider flux as being first order. At linear order, we can, therefore, neglect the $\hat{F}_4 \wedge \hat{F}_4$ term in the equation of motion (3.1.3) and work with the zeroth order metric. The $\hat{F}_4$ equation of motion and the Bianchi identity then simply state that

$$d\hat{F}_4 = d^\dagger \hat{F}_4 = 0 \ , \tag{7.1.6}$$

where the Hodge star is with respect to the zeroth order metric. Inserting the various components (7.1.5) into (7.1.6) we find from the Bianchi identity

$$dG = dJ - G' = dV = dK - V' = 0 \ , \tag{7.1.7}$$

and from the equation of motion

$$d^\dagger G - J' = d^\dagger J = d^\dagger V - K' = d^\dagger K = 0 \ . \tag{7.1.8}$$

Here and elsewhere a prime denotes differentiation with respect to y and the operators $d, d^\dagger$ are now taken with respect to the internal space M_7 with Ricci-flat metric $g^{(0)}$. To summarise, the most general flux is described by a four-form G, a three-form J, a one-form V and a scalar K on the internal space M_7 which are subject to the equations (7.1.7) and (7.1.8).

7.1.5 Spinor Ansatz

A somewhat delicate point in computations of the Killing spinor equations is to find the most general Ansatz for the supersymmetry spinor ϵ. Sometimes, solutions to the Killing spinor equations can be missed if, for example, a simple product Ansatz for ϵ is used. We will, therefore, spend some time discussing this Ansatz for the spinor and finding its most general structure. All relevant conventions for spinors and gamma matrices in the various dimensions involved are collected in the appendix on page 165.

The first point to note is that ϵ must be Majorana, that is

$$\epsilon^c = \epsilon \ . \tag{7.1.9}$$

The conjugation is defined in the Appendix. In general any such spinor can be written in terms of a Dirac spinor ψ like

$$\epsilon = \psi + \psi^c \ . \tag{7.1.10}$$

If a pair of projectors P_+, P_- can be found such that

$$(P_\pm)^2 = P_\pm \ , \quad (P_\pm\psi)^c = P_\mp\psi^c \ , \quad P_+ + P_- = \mathbb{1}_{32} \ , \tag{7.1.11}$$

then we may further write

$$
\begin{aligned}
\epsilon &= \psi + \psi^c \\
&= P_+(\psi + \psi^c) + P_-(\psi + \psi^c) \\
&= (P_+\psi + (P_-\psi)^c) + ((P_+\psi)^c + P_-\psi) \\
&=: \zeta + \zeta^c \,,
\end{aligned} \tag{7.1.12}
$$

where $\zeta = P_+\zeta$. Normally, $P_\pm$ would project onto positive and negative chiralities, but there is no chirality operator in eleven dimensions so we do not yet have a physical interpretation of the manipulation above. However, when we decompose the spinor ζ as

$$\zeta = \xi_+ \otimes \chi \,, \tag{7.1.13}$$

where ξ_+ and χ are four- and seven-dimensional spinors respectively, we can define a sensible pair of projectors by

$$P_\pm := \frac{1}{2}(1 \pm \gamma) \otimes \mathbb{1}_8 \,. \tag{7.1.14}$$

This amounts to imposing that ξ_+ is a positive chirality Weyl spinor, that is, $\xi_+ = \gamma\xi_+$. Its charge conjugate $\xi_- := \xi^c$ is then a negative chirality spinor satisfying $\xi_- = -\gamma\xi_-$. It is possible to show that, for our conventions, we have

$$\gamma^y \xi_+ = -(\xi_-)^* \,, \quad \gamma^y \xi_- = (\xi_+)^* \,. \tag{7.1.15}$$

For an arbitrary complex number z we have $z^* = e^{-2i\arg(z)}z$. Of course, such a result will not in general hold for a multi-component complex object like ξ_+. However, it turns out, after solving the Killing spinor equations, there is no loss of generality in assuming that ξ_+ indeed does satisfy such a relation. Consequently, we introduce a parameter θ such that

$$\gamma^y \xi_\pm = e^{\pm i\theta} \xi_\mp \,. \tag{7.1.16}$$

Note that the internal part χ of the spinor remains unconstrained by the projection, and at zeroth order in the flux will simply be the covariantly constant spinor on the G_2 manifold, η.

We now consider the perturbation of the spinor to linear order. For the four-dimensional spinor we introduce a complex parameter δ such that

$$\xi = (1+\delta)\xi_+ \,, \tag{7.1.17}$$

is the first order four-dimensional spinor. Similarly to the argument about θ above, this is not the most general perturbation of a multi-component complex object, but there will be no loss of generality later if we take ξ to be of the above form. We now use the results of section 5.1.3 to see that the most general linear perturbation of the seven-dimensional spinor η is given by

$$\chi = (1-v_0)\eta + v^A \eta_A \,. \tag{7.1.18}$$

Here v_0, v^A are complex variables parameterising χ. In the full eleven-dimensional picture, note that δ and v_0 are not really independent degrees of freedom since we can absorb δ into a new parameter $v_\delta := \delta - v_0$. We note here that this parameter encodes information about the variation of θ, since basic manipulation of the first order spinor gives

$$\partial\theta = 2\mathrm{Im}(\partial v_\delta) \,. \tag{7.1.19}$$

Because our conventions for the Dirac matrices allow us to express eleven-dimensional charge conjugation as

$$\zeta^c = \xi^c \otimes \chi^c \,, \tag{7.1.20}$$

our final Ansatz for ϵ is then

$$\epsilon = \xi_+ \otimes \left((1+v_\delta)\chi_0 + v^A\chi_A\right) + \xi_- \otimes \left((1+v_\delta^*)\chi_0^c + (v^A)^*\chi_A^c\right) \,. \tag{7.1.21}$$

7.1.6 Bosonic equations

Using the results above, together with the conventions of section 9.1 for the Dirac matrices and of section 5.1 for the action of these matrices on the G_2 spinor, (3.1.5) leads to a set of bosonic first-order equations. They constitute the main formal result in this chapter and are given by

$$
\begin{aligned}
J \lrcorner \varphi &= 12K\cos\theta + \frac{1}{4}G\lrcorner\Phi\sin\theta \ , && (7.1.22)\\
12V\cos\theta &= -\varphi\lrcorner G - J\lrcorner\Phi\sin\theta \ , && (7.1.23)\\
\partial_y\alpha &= \frac{1}{144}G\lrcorner\Phi\cos\theta - \frac{1}{3}K\sin\theta \ , && (7.1.24)\\
d\alpha &= \frac{1}{36}J\lrcorner\Phi\cos\theta - \frac{1}{3}V\sin\theta \ , && (7.1.25)\\
\partial_y v_\delta &= \frac{1}{288}\left(e^{-i\theta}G\lrcorner\Phi - 8iJ\lrcorner\varphi - 48ie^{-i\theta}K\right) \ , && (7.1.26)\\
d\beta &= 2\partial_y\left(\mathrm{Re}(v)\sin\theta + \mathrm{Im}(v)\cos\theta\right) \\
&\quad -\frac{1}{18}J\lrcorner\Phi\cos\theta + \frac{1}{6}V\sin\theta \ , && (7.1.27)\\
\partial_y\begin{pmatrix} \mathrm{Re}(v)\cos\theta \\ +\mathrm{Im}(v)\sin\theta \end{pmatrix} &= \frac{1}{72}\left(\varphi\lrcorner G - 2J\lrcorner\Phi\sin\theta - 6V\cos\theta\right) \ , && (7.1.28)\\
dv_\delta &= \frac{1}{72}\left(2i\varphi\lrcorner G + e^{-i\theta}J\lrcorner\Phi - 12ie^{-i\theta}V\right) \ , && (7.1.29)
\end{aligned}
$$

$$
\begin{aligned}
&4\nabla_A v_B + ie^{-i\theta}\partial_y h_{AB} \\
&-\nabla_C h_{AD}\varphi^{CD}{}_B
\end{aligned}
= \frac{1}{72}\left[\begin{array}{l}
8iG_{ACDE}\Phi^{CDE}{}_B + \frac{i}{5}G^{CDEF}\big(4\Phi_{ACDE}g^{(0)}_{FB} \\
+\Phi_{CDEF}g^{(0)}_{AB} + 12\varphi_{ACD}\varphi_{EFB} + 8\varphi_{CDE}\varphi_{FAB}\big) \\
-24e^{-i\theta}J_{ACD}\varphi^{CD}{}_B - 4e^{-i\theta}J^{CDE}(3\varphi_{ACD}g^{(0)}_{EB} - \\
\varphi_{CDE}g^{(0)}_{AB}) - 24ie^{-i\theta}V^C\varphi_{ABC} + 24Kg^{(0)}_{AB}
\end{array}\right] . \qquad (7.1.30)
$$

These equations link the parameters (α, β, h), associated with the metric, to the various flux components (G, J, V, K) and the quantities (v_δ, v^A, θ) which parameterise the Killing spinor. We note that the y-derivative of β is unconstrained by these equations, which is as we would expect from a residual gauge degree of freedom after the choice of Ansatz. A solution to these first-order partial differential equations preserves two real supercharges ($\mathcal{N} = \frac{1}{2}$ from a four-dimensional point of view) and represents a warped product of a deformed G_2 space and a domain wall in four-dimensional space-time.

7.1.7 Ricci flatness

It is a general result [132, 133] that the integrability of the Killing spinor equation together with the field equations implies that the Einstein equations hold. To linear order in flux, this implies that our solutions should be Ricci-flat. Let us confirm this by explicitly computing the components of the Ricci tensor. Using the Ansatz (7.1.2) we find

$$
\begin{aligned}
R_{\mu\nu} &= (\partial_y^2\alpha + \nabla_A^2\alpha)\eta_{\mu\nu} \\
&= \left(\frac{-1}{144}(dJ - G')\lrcorner\Phi\cos\theta - 13(K' - d^\dagger V)\sin\theta\right)\eta_{\mu\nu}\ , && (7.1.31)\\
R_{yy} &= 3\partial_y^2\alpha + \nabla_A^2\beta + \frac{1}{2}\partial_y^2 h^A{}_A \\
&= \frac{1}{72}(dJ - G')\lrcorner\Phi\cos\theta + \frac{1}{6}(K' - d^\dagger V)\sin\theta\ , && (7.1.32)\\
R_{AB} &= \nabla^A\nabla_B(3\alpha+\beta) + \frac{1}{2}\partial_y^2 h_{AB} + \nabla_A\nabla_{[B}h^C{}_{C]} + \nabla^C\nabla_{[C}h_{B]A} - \frac{1}{2}h^C{}_D R^D{}_{ABC} \\
&= \frac{-1}{72}(dJ - G')\lrcorner\Phi g^{(0)}_{AB}\cos\theta + \frac{1}{3}(K' - d^\dagger V)g^{(0)}_{AB}\sin\theta \\
&\quad + \frac{1}{6}(dJ - G')_{(A}{}^{CDE}\Phi_{B)CDE}\ , && (7.1.33)\\
R_{Ay} &= \partial_y(3\nabla_A\alpha + \nabla_{[A}h^B{}_{B]}) \\
&= -\frac{1}{72}(dG)_{BCDEF}\varphi_A{}^{BC}\varphi^{DEF} - \frac{1}{12}(dV)_{BC}\varphi^{BC}{}_A\ . && (7.1.34)
\end{aligned}
$$

With the first-order relations (7.1.22)–(7.1.30), together with the conditions on the flux (7.1.8) and (7.1.7), we find that these components of the Ricci tensor indeed vanish.

7.2 Explicit eleven-dimensional solution

We now turn to the problem of integrating the bosonic equations (7.1.22)–(7.1.30). Since we are dealing with the case of a general G_2 manifold, this solution will take the form of a sum over basis sets of forms on the manifold. Although we have written the bosonic equations above in the 'raw' form in which they are obtained, there is a certain amount of hidden gauge symmetry that we would like to fix in our solution before we write down such an expansion. In this section, we also consider the zero-mode regime and its relation to compactification.

7.2.1 Simplifying the spinor Ansatz

Before we write down a solution to the eleven-dimensional equations, we reconsider the spinor Ansatz. Our seven-dimensional spinor should be invariant under $SO(7)$ transformations of the tangent space. Using the results of section 5.1.3 we can write such transformations as

$$\chi \mapsto e^{\theta^{AB}\Sigma_{AB}}\chi = e^{\nu^A f_A + \mu^{AB}\rho_{AB}}\chi \ , \tag{7.2.1}$$

where Σ_{AB} are taken in the spinor representation of $SO(7)$ and decompose into f_A, ρ_{AB} under G_2 and θ^{AB}, ν^A and μ^{AB} are real parameters. To first order, this transformation reads

$$\chi \mapsto (1 - v_0)\eta + v^A\eta_A + \nu^A\eta_A \ , \tag{7.2.2}$$

and means that we can 'gauge away' $\mathrm{Re}(v^A)$. The effects of a general coordinate transformation on χ are similar but will in general yield weaker conditions on v^A.

A further point to note is that, since we have a Killing spinor, we are able to form bilinears in this spinor that will be globally defined. In particular, the global vector $w^I = \bar{\epsilon}\Gamma^I\epsilon$, formed in this way should itself be Killing [132]. At linear order, the transverse components of this vector are

$$w^y = \cos\theta + 2(\mathrm{Im}(v_\delta)\sin\theta - \mathrm{Re}(v_\delta)\cos\theta) \ , \qquad w^A = 4\,\mathrm{Im}(v^A) \ . \tag{7.2.3}$$

Since this vector must be Killing, we can then impose

$$\nabla_{(A}v_{B)} = 0 \ , \qquad \partial_y \mathrm{Im}(v_A) = \frac{1}{2}\nabla_A(\mathrm{Re}(v_\delta)\cos\theta - \mathrm{Im}(v_\delta)\sin\theta) \ . \tag{7.2.4}$$

These relations allow us to eliminate v^A from the Killing spinor equation for β, which then takes the form

$$d\beta = -\frac{1}{36}\varphi\lrcorner G\sin\theta\cos\theta - \frac{3}{72}J\lrcorner\Phi\cos\theta + \frac{1}{6}V\sin\theta \ . \tag{7.2.5}$$

7.2.2 Simplifying the relations for metric perturbations

We also make a gauge choice for h_{AB}, by putting it in the standard 'harmonic gauge' so that

$$\nabla_B h^B{}_A = \frac{1}{2}\nabla_A \mathrm{tr}(h) \ . \tag{7.2.6}$$

Our result (7.1.22)–(7.1.30) can be simplified considerably by splitting into real and imaginary parts, projecting out the irreducible G_2 representations associated with the two free indices, using the simplifications of the spinor Ansatz as above and making the harmonic gauge choice for h. We are then able to derive the following set of physically equivalent first order relations

$$\begin{aligned}
J\lrcorner\varphi &= 21K\cos\theta + \frac{5}{8}G\lrcorner\Phi\sin\theta \ , && (7.2.7)\\
\partial_y \mathrm{tr}(h) &= -\frac{5}{72}G\lrcorner\Phi\cos\theta + \frac{7}{3}K\sin\theta \ , && (7.2.8)\\
d\,\mathrm{tr}(h) &= 4V\sin\theta + \frac{1}{3}J\lrcorner\Phi \ , && (7.2.9)\\
\begin{bmatrix} \partial_y(P_{27}h)_{AB} - \\ \nabla_C(P_{27}h)_{D(A}\varphi^{CD}{}_{B)}\sin\theta \end{bmatrix} &= -\frac{1}{6}(P_{27}G)_{(A}{}^{CDE}\Phi_{B)CDE}\cos\theta \ , && (7.2.10)\\
\nabla_C(P_{27}h)_{D(A}\varphi^{CD}{}_{B)}\cos\theta &= -\frac{1}{6}(P_{27}G)_{(A}{}^{CDE}\Phi_{B)CDE}\sin\theta \\
&\quad +\frac{1}{2}(P_{27}J)_{(A}{}^{CD}\varphi_{B)CD} \ , && (7.2.11)
\end{aligned}$$

where the projector P_{27} projects out the **27** representation in the G_2 decomposition of the various tensors, as explained in section 5.1.2.

7.2.3 Zero-mode regime

The field equations (7.1.7) and (7.1.8) imply that

$$\begin{aligned}
\Delta_7 G = G'' \ , \quad \Delta_7 J = J'' \ , \\
\Delta_7 V = V'' \ , \quad \Delta_7 K = K'' \ ,
\end{aligned} \tag{7.2.12}$$

where Δ_7 is the seven-dimensional Laplacian with respect to the zeroth order metric $g^{(0)}$. We call solutions for which both sides of these equations are zero the 'zero modes' and those for which both are equal to a non-zero constant the 'massive modes'.

The physical reasoning behind this is that of chapter 4: operators like Δ_7 will be associated with the inverse of the radius of compactification of M_7. When this is reduced down to small scales, this makes Δ_7 produce extremely large constant non-zero eigenvalues, which are effective masses in the four-dimensional theory. Since these masses will typically be at the Planck scale, they can be ignored in constructing the four-dimensional effective theory, and so the zero-mode regime is of particular interest to us.

We firstly note that on G_2 manifolds there are no harmonic one-forms, and so the following terms in the flux vanish in the zero-mode regime:

$$\varphi \lrcorner G = J \lrcorner \Phi = V = 0 \; . \tag{7.2.13}$$

This also constrains the spinor so that $v^A = 0$, since otherwise the equation (7.1.30) would make v^A a harmonic one-form. Such constraints on seven-dimensional vectors mean that in the zero-mode regime, using the first-order bosonic equations (7.1.22)–(7.1.30) we have

$$d\alpha = d\beta = 0 \; . \tag{7.2.14}$$

A similar argument to that for the flux can be made for the graviton h_{AB}, which from (7.1.33), (7.2.6) and (7.2.14) obeys

$$\Delta_L h = -\partial_y^2 h \; , \tag{7.2.15}$$

where

$$(\Delta_L h)_{AB} := \nabla_C \nabla^C h_{AB} + 2R^C{}_{(AB)D} h^D{}_C \; . \tag{7.2.16}$$

In this case, we also argue that Δ_L will be associated with a Planck-scale effective mass upon compactification and so can be ignored.

The arguments above allow us to write a 'zero-mode' version of the first-order eleven-dimensional bosonic equations (7.1.22)–(7.1.30)

$$\begin{aligned}
\partial_y \alpha &= \frac{1}{144} G \lrcorner \Phi \cos\theta - \frac{1}{3} K \sin\theta \ , && (7.2.17)\\
\partial_y \mathrm{tr}(h) &= -\frac{5}{72} G \lrcorner \Phi \cos\theta + \frac{7}{3} K \sin\theta \ , && (7.2.18)\\
\partial_y (P_{27} h)_{AB} &= -\frac{1}{6} (P_{27} G)_{(A}{}^{CDE} \Phi_{B)CDE} \cos\theta \ , && (7.2.19)\\
\partial_y \theta &= -\frac{1}{48} G \lrcorner \Phi \sin\theta - K \cos\theta \ , && (7.2.20)\\
P_{27} J &= -P_{27} \star G \ . && (7.2.21)
\end{aligned}$$

We shall use these equations later to compare with the bosonic equations that we derive from the four-dimensional Killing spinor equations.

7.2.4 Fourier expansion of the flux

We shall now expand each component of the flux as a sum over forms on the G_2 manifold M_7. At the zero-mode level, this expansion involves the harmonic forms and is given by

$$\begin{aligned}
G_0 &= \textstyle\sum_i G_i \widetilde{\pi}^i \ , & V_0 &= 0 \ , \\
J_0 &= \textstyle\sum_i J_i \star \widetilde{\pi}^i \ , & K_0 &= \mathrm{const.}
\end{aligned} \qquad (7.2.22)$$

Here G_i, J_i and K_0 are constants, and we have introduced a set of harmonic four-forms on M_7, $\{\widetilde{\pi}^i\}_{i=1}^{b_3(M_7)}$ where $b_3(M_7)$ is the 3rd Betti number of M_7. Notationally, we will sometimes adopt implicit summation over i, j-type indices but leave them in for clarity at present.

In the massive regime, the expansion is slightly more complicated, since we must introduce a further set of massive four-forms on M_7, $\{\widetilde{\Pi}^n\}$ satisfying

$$\Delta_7 \widetilde{\Pi}^n = (m_n)^2 \widetilde{\Pi}^n \ . \qquad (7.2.23)$$

We can then use the Hodge star to construct a set of 3-forms, $\{\star\widetilde{\Pi}^n\}$, with

$$\Delta_7 \star \widetilde{\Pi}^n = (m_n)^2 \star \widetilde{\Pi}^n \ . \qquad (7.2.24)$$

Then the massive modes of G, J can be expanded in terms of these forms, leading to

$$G_{\rm massive} = \sum_n G_n(y)\widetilde{\Pi}^n \ , \quad J_{\rm massive} = \sum_n J_n(y) \star \widetilde{\Pi}^n \ , \tag{7.2.25}$$

with y-dependent expansion coefficients G_n and J_n. The equations of motion for the flux then imply

$$\begin{aligned} G_n'' &= (m_n)^2 G_n \quad \Rightarrow \quad G_n(y) = G_n^+ e^{m_n y} + G_n^- e^{-m_n y} \ , \\ J_n'' &= (m_n)^2 J_n \quad \Rightarrow \quad J_n(y) = J_n^+ e^{m_n y} + J_n^- e^{-m_n y} \ , \end{aligned} \tag{7.2.26}$$

for constant $G_n^+, G_n^-, J_n^+, J_n^-$. The massive expansion of K and V can be done in a similar way. We can write both in terms of a set of functions $\{f^p\}$ obeying $\Delta_7 f^p = (M_p)^2 f^p$ so that

$$\begin{aligned} V_{\rm massive} &= \sum_p \frac{1}{M_p}\left(V_p^+ e^{M_p y} + V_p^- e^{-M_p y}\right) df^p \ , \\ K_{\rm massive} &= \sum_p \left(K_p^+ e^{M_p y} + K_p^- e^{-M_p y}\right) f^p \ , \end{aligned} \tag{7.2.27}$$

for constant V_p^+, V_p^-, K_p^+ and K_p^-. We have introduced a factor of M_p in the first of these relations to compensate for the mass associated with the exterior derivative d. To see why this is the correct expansion for our solution consider the linear equation for $d\alpha$, (7.1.25),

$$\begin{aligned} d\alpha &= \frac{1}{36} J \lrcorner \Phi \cos\theta - \frac{1}{3} V \sin\theta \\ &= \frac{1}{36}\sum_n (J_n^+ e^{m_n y} + J_n^- e^{-m_n y})(\star\widetilde{\Pi}^n)\lrcorner\Phi\cos\theta \\ &\quad -\frac{1}{3}\sum_p \left(V_p^+ e^{M_p y} + V_p^- e^{-M_p y}\right)(df^p)\sin\theta \ . \end{aligned} \tag{7.2.28}$$

From the nilpotency of d this equation implies that the right hand side of this equation is a closed 1-form and thus can be written uniquely as the sum of an exact 0-form and a harmonic 1-form. As there are no harmonic 1-forms on M_7 due to G_2 holonomy, each term on the right hand side must be exact. This justifies our expansion of V in terms of the $\{df^p\}$, and also allows us to define a further set of functions by

$$d\Pi^n_{(0)} := m_n(\star\widetilde{\Pi}^n)\lrcorner\Phi \ . \tag{7.2.29}$$

7.2.5 Integration of the bosonic equations

After expanding as above, the direct integration of the bosonic equations (7.1.22)–(7.1.30) is relatively straightforward, particularly as we can take θ as a constant to linear order. Performing this integration then leads the following complete solution for the metric components

$$\begin{aligned}
\alpha(y,x^A) &= \left(\frac{-1}{3}K_0 y - \frac{1}{3}\sum_p \frac{1}{m_p}\left((K_p^+ + V_p^+)e^{m_p y} - (K_p^- - V_p^-)e^{-m_p y}\right) f^p\right)\sin\theta \\
&\quad + \frac{1}{144}\left(\sum_i G_i \widetilde{\pi}^i y + \sum_n \frac{1}{m_n}(G_n^+ e^{m_n y} - G_n^- e^{-m_n y})\widetilde{\Pi}^n\right) \lrcorner \Phi \cos\theta \\
&\quad + \frac{1}{36}\sum_n \frac{1}{m_n}\left(J_n^+ e^{m_n y} + J_n^- e^{-m_n y}\right)\Pi^n_{(0)}\cos\theta + \alpha_0 \ , \qquad (7.2.30) \\
\beta(y,x^A) &= \frac{1}{36}\left(\sum_i G_i \widetilde{\pi}^i y + \sum_n \frac{1}{m_n}(G_n^+ e^{m_n y} - G_n^- e^{-m_n y})\widetilde{\Pi}^n\right) \lrcorner \Phi \sin\theta\cos\theta \\
&\quad + \frac{1}{6}\sum_p \frac{1}{m_p}\left(V_p^+ e^{m_p y} + V_p^- e^{-m_p y}\right) f^p \sin\theta \\
&\quad - \frac{3}{72}\sum_n \frac{1}{m_n}(J_n^+ e^{m_n y} + J_n^- e^{-m_n y})\Pi^n_{(0)}\cos\theta + \beta_0 \ , \qquad (7.2.31) \\
\mathrm{tr}(h)(y,x^A) &= \left(\frac{7}{3}K_0 y + \sum_p \frac{1}{m_p}\left(\left(\frac{7}{3}K_p^+ + 4V_p^+\right)e^{m_p y}\right.\right. \\
&\quad \left.\left. - \left(\frac{7}{3}K_p^- - 4V_p^-\right)e^{-m_p y}\right) f^p\right)\sin\theta \\
&\quad - \frac{5}{72}\left(\sum_i G_i \widetilde{\pi}^i y + \sum_n \frac{1}{m_n}(G_n^+ e^{m_n y} - G_n^- e^{-m_n y})\widetilde{\Pi}^n\right) \lrcorner \Phi \cos\theta \\
&\quad + \frac{1}{3}\sum_n \frac{1}{m_n}(J_n^+ e^{m_n y} + J_n^- e^{-m_n y})\Pi^n_{(0)}\cos\theta + h_0 \ . \qquad (7.2.32)
\end{aligned}$$

Here, α_0, β_0, h_0 are constants of integration and we have gauged away the y-dependence of β. Note that the first term, proportional to y, on each right-hand side represents the zero-mode contributions while all other terms correspond to heavy modes. The above expressions make it clear that the massive modes are proportional to the inverse effective mass, and so upon compactification we expect them to be negligible. This means, we should consider only the zero mode regime when we come to the four-dimensional effective theory later.

For the traceless part of the metric, the integration is slightly less straightforward, but proceeds along similar lines so that at zero mode

$$(P_{27}h_{\text{zero-mode}})_{AB} = -\frac{1}{6}\sum_i G_i y\cos\theta (P_{27}\widetilde{\pi}^i)_{(A}{}^{CDE}\Phi_{B)CDE} \ . \tag{7.2.33}$$

The massive modes can be obtained by solving the equations

$$\begin{aligned}
&\partial_y(P_{27}h_{\text{massive}})_{AB} - \nabla_C(P_{27}h_{\text{massive}})_{D(A}\varphi^{CD}{}_{B)}\sin\theta = \\
&\quad -\frac{1}{6}\sum_n \left(G_n^+ e^{m_n y} + G_n^- e^{-m_n y}\right)(P_{27}\widetilde{\Pi}^n)_{(A}{}^{CDE}\Phi_{B)CDE}\cos\theta \ , \\
&\nabla_C(P_{27}h_{\text{massive}})_{D(A}\varphi^{CD}{}_{B)}\cos\theta = \\
&\quad -\frac{1}{6}\sum_n \left((G_n^+\sin\theta + J_n^+)e^{m_n y} + (G_n^-\sin\theta + J_n^-)e^{-m_n y}\right)(P_{27}\widetilde{\Pi}^n)_{(A}{}^{CDE}\Phi_{B)CDE} \ ,
\end{aligned} \tag{7.2.34}$$

which can be done explicitly.

7.2.6 Curvature singularities

Analogously to the calculation in [87], we now look for the curvature singularities that correspond to vanishing volume of the compact space. This quantity is given by

$$\mathcal{V} := \text{Vol}(M_7) = \int_{M_7}\sqrt{g}\, d^7x \ . \tag{7.2.35}$$

We can then differentiate this to linear order using the relation (7.2.18) so that

$$\begin{aligned}
\partial_y\mathcal{V} &= \int_{M_7}\frac{1}{2}(g^{(0)})^{AB}\partial_y h_{AB}\sqrt{g^{(0)}}d^7x \\
&= \int_{M_7}\left(-\frac{5}{144}G\lrcorner\Phi\cos\theta + \frac{7}{6}K\sin\theta\right)\sqrt{g^{(0)}}d^7x \\
&= \int_{M_7}\left(-\frac{5}{6}\varphi\wedge G\cos\theta + \frac{7}{6}\star K\sin\theta\right) \ .
\end{aligned} \tag{7.2.36}$$

We can then use the equations of motion for the flux (7.1.7) and (7.1.8) to show that

$$\partial_y^2\mathcal{V} = \int_{M_7} d\left(-\frac{5}{6}(\varphi\wedge J)\cos\theta + \frac{7}{6}(\star V)\sin\theta\right) = 0 \ , \tag{7.2.37}$$

since M_7 has no boundary. The volume must thus depend linearly on the coordinate y, and so it must be zero for some value of y, which will correspond to a curvature singularity of the internal space. Of course, as the volume of the compact space becomes small, we are no longer entitled to use simply eleven-dimensional supergravity as our theory. We might also reasonably expect that although the linear terms in flux in (7.2.37) vanish, the higher-order contributions may not.

7.3 Inclusion of brane sources

In general, we expect $(p+1)$-form fields to be sourced by an extended charged p-brane. In M-theory, there are two sensible choices for p, given that the only form field present is the three-form field A. These are the 'fundamental' membrane (M2-brane) and the 'magnetic' five-brane (M5-brane).

We shall consider each of these in turn, as both may well support the kind of domain wall solution that we are considering. In the case of the M2-brane, this would happen by its sitting in the external space, whereas the M5-brane would have to wrap a three-cycle in the compact space in an appropriate way.

Our approach shall be to solve the brane equations of motion for each system, and then try to match these solutions to appropriate specialisations of the bulk solution that we have so far been considering. In doing this, we will find that the inclusion of a brane source fixes the value of θ in such a way that our bulk solution can either support the M2-brane or the M5-brane but not both.

We further find that in each case the brane splits the y-direction into two regions, each with different values of the flux and with a 'jump' in certain components of the flux across the brane proportional to the brane tension.

7.3.1 General brane action in M-theory

In this section, we will quote some general results about classical membranes in eleven-dimensional supergravity, following [103]. The easiest to consider is the fundamental membrane, which couples to the 3-form field $\hat{A}_3$ by the action

$$S_2 = T_2 \int_{W_3} d^3\mathcal{X} \left(-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^I \partial_j X^J g_{IJ} - \frac{1}{2}\sqrt{-\gamma} - \frac{1}{4!}\varepsilon^{ijk}\partial_i X^I \partial_j X^J \partial_k X^K (\hat{A}_3)_{IJK} \right) , \tag{7.3.1}$$

where X^I are the brane coordinates, $i, j, \ldots = 0, \ldots, 2$ are brane worldvolume indices and W_3 is the worldvolume of the brane parameterised by the coordinates $\mathcal{X}^i$. This membrane couples to our bulk supergravity action S_{11} in (3.1.1) to produce the total action

$$S_{\text{Total}} = S_{11} + \int d^{11}x \, S_2 \delta(X - x) \, . \tag{7.3.2}$$

This modifies the previous equations of motion to give

$$d\hat{F}_4 = 0 \, , \tag{7.3.3}$$

$$d^\dagger \hat{F}_4 = -2\mathcal{J}_2 \, , \tag{7.3.4}$$

$$\begin{aligned} \hat{R}_{IJ} &= \frac{1}{12}\left((\hat{F}_4)_{IKLM}(\hat{F}_4)_J{}^{KLM} - \frac{1}{12} g_{IJ} \hat{F}_4 \lrcorner \hat{F}_4 \right) \\ &\quad + \sqrt{-g}\left(T_{IJ} - \frac{1}{9} g_{IJ} T \right) \, . \end{aligned} \tag{7.3.5}$$

The current and stress-energy associated with the brane are

$$\mathcal{J}_2^{IJK} = T_2 \int d^3\mathcal{X} \varepsilon^{ijk} \partial_i X^I \partial_j X^J \ldots \partial_k X^K \frac{\delta^{11}(x - X)}{\sqrt{-g}} \, , \tag{7.3.6}$$

$$T^{IJ} = -T_2 \int d^3\mathcal{X} \sqrt{-\gamma}\gamma^{ij} \partial_i X^I \partial_j X^J \frac{\delta^{11}(x - X)}{\sqrt{-g}} \, , \tag{7.3.7}$$

and the membrane worldvolume equations of motion are

$$\begin{aligned} \partial_i \left(\sqrt{-\gamma}\gamma^{ij} \partial_j X^J g_{IJ} \right) &= \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^J \partial_j X^K \partial_I g_{JK} \\ &\quad + \frac{1}{4!}\varepsilon^{ijk}\partial_i X^J \partial_j X^K \partial_k X^L (\hat{F}_4)_{IJKL} \, , \end{aligned} \tag{7.3.8}$$

$$\gamma_{ij} = \partial_i X^I \partial_j X^J g_{IJ} \, . \tag{7.3.9}$$

The coupling of the five-brane to the bulk action is slightly more subtle, and to be done properly requires the approach of [105]. As we are only interested in the effect of the five-brane on the equations of motion rather than the duality-symmetric formulation of the low-energy M-theory action, we merely state here the five-brane equations of motion when we set the worldvolume two-form to zero. Analogously to the magnetic monopole in electrodynamics, the five-brane sources to the dual of the flux, $\star\hat{F}_4$, in the same way that the membrane couples to $\hat{F}_4$. We then have bulk equations of motion

$$d^\dagger \hat{F}_4 = 0 \,, \tag{7.3.10}$$

$$d\hat{F}_4 = 2\star \mathcal{J}_5 \,, \tag{7.3.11}$$

$$\begin{aligned} \hat{R}_{IJ} &= \frac{1}{12}\left((\hat{F}_4)_{IKLM}(\hat{F}_4)_J{}^{KLM} - \frac{1}{12}g_{IJ}\hat{F}_4 \lrcorner \hat{F}_4\right) \\ &\quad + \sqrt{-g}\left(T_{IJ} - \frac{1}{9}g_{IJ}T\right) \,. \end{aligned} \tag{7.3.12}$$

In this case the current and stress-energy are

$$\mathcal{J}_5^{I_1\ldots I_6} = T_5 \int d^6\mathcal{X}\varepsilon^{i_1\ldots i_6}\partial_{i_1}X^{I_6}\ldots\partial_{i_6}X^{I_6}\frac{\delta^{11}(x-X)}{\sqrt{-g}} \,, \tag{7.3.13}$$

$$T^{IJ} = -T_5 \int d^6\mathcal{X}\sqrt{-\gamma}\gamma^{ij}\partial_i X^I \partial_j X^J \frac{\delta^{11}(x-X)}{\sqrt{-g}} \,, \tag{7.3.14}$$

with worldvolume equations

$$\begin{aligned} \partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_j X^J g_{IJ}\right) &= \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^J \partial_j X^K \partial_I g_{JK} \\ &\quad + \frac{1}{7!}\varepsilon^{i_1\ldots i_6}\partial_{i_1}X^{I_6}\ldots\partial_{i_6}X^{I_6}(\star\hat{F}_4)_{I_1\ldots I_6} \,, \end{aligned} \tag{7.3.15}$$

$$\gamma_{ij} = \partial_i X^I \partial_j X^J g_{IJ} \,. \tag{7.3.16}$$

We shall now go on to apply these results for the M2-brane and M5-brane in the context of our existing bulk solution, noting that from the brane equations the flux is naturally of the order of the brane tension, which we must bear in mind when we truncate our results to linear order.

We will then solve the membrane equations of motion at linear order, but only considering the zero-mode part, which will involve 'smoothing over' the seven-dimensional part of the delta functions in the action, essentially for reasons of simplicity. There is no clear reason why a fuller treatment of the massive modes should not yield fundamentally the same conclusions as below.

7.3.2 Fundamental membrane

It is natural to consider the membrane as supporting a domain wall solution by placing it in the external space with its transverse coordinate along the special direction, previously called y, of the solution. We implement this with the following 'static gauge' choice for the membrane coordinates

$$X^\mu = \xi^\mu \ , \qquad X^y = \text{const.} \ , \qquad X^A = \text{const.} \tag{7.3.17}$$

If we then consider the linearised eleven-dimensional equations of motion, taking only the zero mode of the internal space part, we find the following modifications

$$\begin{aligned} \partial_y^2 \alpha &= \frac{-2}{3} T_2 \delta(y) \ , & (7.3.18) \\ \partial_y^2 \text{tr}(h) &= \frac{14}{3} T_2 \delta(y) \ , & (7.3.19) \\ K' &= 2T_2 \delta(y) \ . & (7.3.20) \end{aligned}$$

The membrane worldvolume equations are

$$\partial_y \alpha = -\tfrac{1}{3} K \ , \qquad d\alpha = -\tfrac{1}{3} V \ , \tag{7.3.21}$$

and imposing worldvolume supersymmetry (κ-symmetry) gives the following condition:

$$\widetilde{P_-} \epsilon = 0 \ , \tag{7.3.22}$$

where $\widetilde{P_\pm} := \frac{1}{2}(1 \pm i\gamma^y \gamma) \otimes \mathbb{1}$ can be interpreted as projecting out different components of eight-dimensional chirality. This condition turns out to be equivalent to taking $\sin\theta = 1$ in the Killing spinor Ansatz.

The physical interpretation of this solution is shown in Figure 7.1. Having smoothed over the compact space, we can consider the remaining four-dimensional space to be split into two regions by the membrane, each containing different (constant) values for the flux.

Suppose we write the flux in Region I as $K_{\rm I}$ and in Region II as $K_{\rm II}$, then we can integrate the equations (7.3.18)-(7.3.20) by writing

$$K(y) = 2T_2(K_{\rm II} - K_{\rm I})\Theta(y) + K_{\rm I} \ , \tag{7.3.23}$$

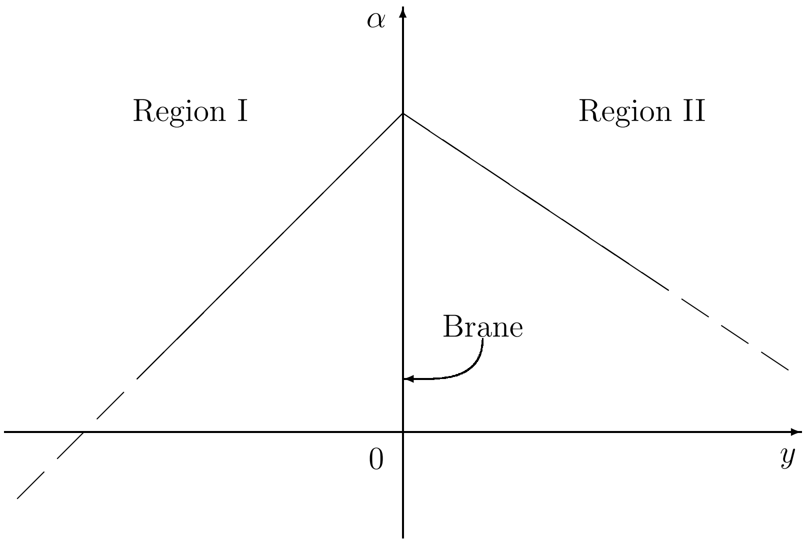

Figure 7.1: Physical interpretation of the membrane solution showing warp factor α against transverse coordinate y.

$$\partial_y \alpha = -\tfrac{1}{3}K \ , \qquad \partial_y \mathrm{tr}(h) = \tfrac{7}{3}K \ , \tag{7.3.24}$$

where Θ is the step function defined by

$$\Theta(x) := \begin{cases} 0 \text{ for } x < 0 \ , \\ \frac{1}{2} \text{ for } x = 0 \ , \\ 1 \text{ for } x > 0 \ . \end{cases} \tag{7.3.25}$$

This solution is consistent with taking the $\sin\theta = 1$ specialisation of the zero-mode bosonic equations (7.1.22)–(7.1.30) above in each of the bulk regions. The 'jump' in flux from one region to another is proportional to the membrane tension. It is worth noting that membranes could be stacked to make this jump proportional to the number of membranes times the tension.

7.3.3 Magnetic five-brane

The situation for the M5-brane is slightly more complicated than for the M2-brane, given that it has three more worldvolume dimensions than the domain wall. For this reason, three of the worldvolume dimensions should be wrapped on some three-cycle

Σ_3 with in the G_2 space M_7. For the solution that arises from this configuration to be supersymmetric, this cycle must be *calibrated* by the G_2 three-form φ, which means that

$$\mathrm{Vol}(\Sigma_3) = \int_{\Sigma_3} \varphi \ . \tag{7.3.26}$$

Three-forms that are calibrated by φ are called *associative* cycles, and four-forms calibrated by Φ are called *co-associative*. The appropriate choice for the brane coordinates is given by

$$X^\mu = \xi^\mu \ , \qquad X^y = \text{const.} \ , \qquad X^a = \sigma^a \ , \qquad X^{\tilde{A}} = \text{const.} \ , \tag{7.3.27}$$

where we let σ^a be the coordinates of Σ_3 and use the indices $\tilde{A}$ to denote directions perpendicular to Σ_3. Considering the equations of motion for this system in the zero mode regime then gives

$$\begin{aligned} \partial_y^2 \alpha &= \frac{-1}{3} T_5 \delta(y) \ , & (7.3.28)\\ \partial_y^2 \mathrm{tr}(h) &= \frac{10}{3} T_5 \delta(y) \ , & (7.3.29)\\ G' &= \frac{2}{7} T_5 \delta(y) \Phi \ , & (7.3.30) \end{aligned}$$

with worldvolume equations

$$6\partial_y \alpha + \frac{3}{7}\partial_y \mathrm{tr}(h) = \frac{-1}{84} G \lrcorner \Phi \ . \tag{7.3.31}$$

Imposing κ-symmetry gives the following condition:

$$P_-^y \epsilon = 0 \ , \tag{7.3.32}$$

where $P_\pm^y := \frac{1}{2}(1 \pm \gamma^y) \otimes \mathbb{1}$ can be interpreted as projecting out different components of 'y-chirality'. This condition turns out to be equivalent to taking $\cos\theta = 1$ in the Killing Spinor Ansatz.

Our four-dimensional picture then looks very similar to that of the membrane, with two separated regions containing different values for the flux, such that

$$G(y) = 2T_5(G_{\mathrm{II}} - G_{\mathrm{I}})\Theta(y) + G_{\mathrm{I}} \ . \tag{7.3.33}$$

Note that only the singlet part of G is shifted by the brane since, from (7.3.30), the difference $G_{\mathrm{II}} - G_{\mathrm{I}}$ is proportional to the G_2 invariant four-form Φ. As a result, we need not consider the traceless part of h and can thus simply integrate (7.3.28) and (7.3.29) to give

$$\partial_y \alpha = \frac{-1}{144} G \lrcorner \Phi \ , \qquad \partial_y \mathrm{tr}(h) = \frac{5}{72} G \lrcorner \Phi \ . \tag{7.3.34}$$

Similarly to the M2-brane case, this is consistent with the $\cos\theta = 1$ specialisation of the bosonic equations (7.1.22)–(7.1.30). We also have the partition of the external space into two separate regions each with different constant values for the flux, with the jump in flux between these regions proportional to the brane tension. In contrast to the M2-brane, however, the relevant component of flux is the singlet of G rather than the K. It will also be possible to stack branes so that the jump in flux is proportional to the number of stacked branes.

Note that although both the membrane and five-brane very naturally couple to our bulk solution, at the order we are considering, the existence of supersymmetric configurations with two real supercharges containing both types of brane is ruled out.

7.4 Four-dimensional effective theory

When eleven-dimensional supergravity is compactified on a G_2 manifold, the effective field theory is given by four-dimensional $\mathcal{N} = 1$ supergravity. In this section, we outline how the quantities in the four-dimensional action are related to the eleven-dimensional quantities. We then present the conditions for a supersymmetric domain wall solution in four dimensions, and integrate these equations. Finally, we uplift the four-dimensional equations to eleven dimensions, and check that the result indeed matches our earlier one, obtained directly from the eleven-dimensional theory.

The action for the four-dimensional theory is given in section 4.1.1, and we use the notation from that section.

7.4.1 Four-dimensional supergravity from M-Theory on a G_2 space

The relationship between the theory above and eleven-dimensional supergravity on a compact G_2 manifold was covered in [15, 49] and discussed above in chapter 5. Throughout this section we use just the zero-mode part of form field strength G which is written in terms of a set of harmonic four-forms $\{\widetilde{\pi}^i\}$ as

$$G = G_i \widetilde{\pi}^i \ , \tag{7.4.1}$$

where we now use implicit summation rather than the explicit notation of Section 7.2. Our first step is to expand both the 3-form field $\hat{A}_3$ and the G_2 3-form φ in terms of a dual set of harmonic three-forms $\{\pi_i\}_{i=1}^{b_3(M_7)}$ obeying

$$\int_{M_7} \pi_i \wedge \widetilde{\pi^j} = \delta_i^j \ , \tag{7.4.2}$$

so that

$$\varphi \ = \ \varphi^i \pi_i \ , \tag{7.4.3}$$

$$\hat{A}_3 \ = \ A^i \pi_i + \mathring{A} \ , \tag{7.4.4}$$

where $G = d\mathring{A}$ in a simplification of (5.2.9), so that $\hat{F}_4 = d\hat{A}_3$ still holds. Detailed consideration of the compactification of the eleven-dimensional theory shows that the φ^i are the metric moduli of the G_2 manifold, M_7, and the moduli A^i appear as axions in the 4-dimensional theory. This means that we can write the superfields as

$$T^i = \varphi^i + iA^i \ . \tag{7.4.5}$$

In our conventions, the superpotential is then

$$W = \frac{7^{3/2}}{4} \int_{M_7} G \wedge T = \frac{7^{3/2}}{4} G_i T^i \ . \tag{7.4.6}$$

We shall now consider the idea from [15, 58] that each term in (7.4.6) is 'sourced' by the wrapped M5-brane and M2-brane respectively. From (3.1.3)—the equation of motion for the flux—it can be shown that

$$\int_{M_7} \left(\star K + \frac{1}{2} A \wedge G \right) = 0 \Rightarrow G_i A^i = -\mathcal{V} K \ , \tag{7.4.7}$$

where $\mathcal{V}$ is the volume of the compact space as defined in (7.2.35). Also, as G is harmonic, the projector of its singlet must be constant over M_7 and so

$$G \lrcorner \Phi = \frac{1}{\mathcal{V}} \int_{M_7} \sqrt{-g} G \lrcorner \Phi \Rightarrow G_i \varphi^i = \frac{1}{24} \mathcal{V} G \lrcorner \Phi \ . \tag{7.4.8}$$

We can thus substitute (7.4.7) and (7.4.8) into (7.4.6) to rewrite the superpotential as

$$W = \frac{7^{3/2}\mathcal{V}}{4} \left(\frac{1}{24} G \lrcorner \Phi - iK \right) \ . \tag{7.4.9}$$

Clearly, this expression contains a term proportional to the singlet of G as well as one proportional to K. As we saw in section 7.3, the wrapped M5-brane acts as a source for the singlet of G and the M2-brane acts as a source for K. Therefore, we can interpret the moduli superpotential for the four-dimensional effective theory associated with M-theory on G_2 manifolds as being sourced by contributions from the wrapped M5-brane and the M2-brane.

The Kähler potential for this theory and its derivatives with respect to the superfields is given by

$$\begin{aligned} \mathcal{K} &= -3 \ln (7\mathcal{V}) \ , & (7.4.10) \\ \Rightarrow \mathcal{K}_i &= \frac{\partial \mathcal{K}}{\partial T^i} = \frac{-1}{2\mathcal{V}} \int_{M_7} \pi_i \wedge \Phi = -2\varphi_i \ , & (7.4.11) \\ \Rightarrow \mathcal{K}_{i\bar{j}} &= \frac{1}{4\mathcal{V}} \int_{M_7} \pi_i \wedge \star \pi_j \ . & (7.4.12) \end{aligned}$$

Using these expressions, we can write G in terms of the dual set of four-forms like

$$G = \frac{1}{4\mathcal{V}} G^i \star \pi_i \ . \tag{7.4.13}$$

This is all the input from eleven dimensions that we need to write down and solve the appropriate four-dimensional Killing spinor equations. We will return to the links between four and eleven dimensions later.

7.4.2 Four-dimensional Killing spinor equations

We shall now set up the conditions for supersymmetric domain wall solutions in the four-dimensional supergravities derived from our eleven-dimensional theory as above. We start with a warped metric Ansatz

$$ds_4^2 = e^{a(z)}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \ , \tag{7.4.14}$$

and z-dependent scalar fields $T^i = T^i(z)$. For such a field configuration the first-order relations [1] derived from the Killing spinor equations are given by [134]

$$\begin{aligned} \partial_z a &= e^{-i\theta} e^{\frac{1}{2}\mathcal{K}} W \ , & (7.4.15)\\ \partial_z T^i &= -e^{i\theta} e^{\frac{1}{2}\mathcal{K}} \mathcal{K}^{i\bar{\jmath}} \ D_{\bar{\jmath}}\overline{W} \ , & (7.4.16)\\ \partial_z \theta &= -\mathrm{Im}\left[(\partial_z T^i)\mathcal{K}_i\right] \ . & (7.4.17) \end{aligned}$$

Here, θ parameterises the four-dimensional Killing spinor in a similar way to the quantity θ we have used to parameterise the earlier eleven-dimensional spinor. When we consider the links between four and eleven dimensions later, we will find that they are in fact the same at zero mode level in eleven dimensions.

Solutions to these equations are BPS states, which preserve half of the maximum number of supercharges possible in the $\mathcal{N} = 1$, $D = 4$ theory, and automatically satisfy its equations of motion. Using the expressions for T^i, W and $\mathcal{K}$ from eleven dimensions above allows us to write the first-order relations as

$$\begin{aligned} \partial_z a &= \frac{\mathcal{V}^{-3/2}}{4}\left(G_i\varphi^i \cos\theta + G_i A^i \sin\theta\right) \ , & (7.4.18)\\ G_i A^i \cos\theta &= G_i \varphi^i \sin\theta \ , & (7.4.19)\\ \partial_z \varphi^i &= \frac{\mathcal{V}^{-3/2}}{4}\left((2G_j\varphi^j\varphi^i - G^i)\cos\theta + 2G_jA^j\varphi^i \sin\theta\right) \ , & (7.4.20)\\ \partial_z A^i &= \frac{\mathcal{V}^{-3/2}}{4}\left((2G_j\varphi^j\varphi^i - G^i)\sin\theta - 2G_jA^j\varphi^i \cos\theta\right) \ , & (7.4.21)\\ \partial_z \theta &= -\frac{\mathcal{V}^{-3/2}}{2} G_i\varphi^i \sin\theta \ . & (7.4.22) \end{aligned}$$

It is also worth noting that the relation

$$\partial_z \mathcal{V} = \frac{\mathcal{V}^{-1/2}}{6}\left(5G_i\varphi^i\cos\theta + 7G_iA^i\sin\theta\right) \tag{7.4.23}$$

[1] There is one further equation that constrains the four-dimensional Killing spinor but it does not really impact on our calculation.

can be derived from (7.4.18)–(7.4.22). This will be important later when we find explicit solutions.

We now consider how to integrate these four-dimensional equations in purely four-dimensional language, before uplifting them to compare with the eleven-dimensional equations.

7.4.3 Integrating the four-dimensional equations

We now present the most general solution to equations (7.4.15)–(7.4.17), which are the conditions for a supersymmetric domain wall to four-dimensional supergravity, given that the supergravity descends from M-theory on a G_2 manifold. The solution can be written, in terms of new moduli fields f^i, as

$$\begin{aligned} a &= \frac{1}{2}\ln|\cot\theta| + C \ , & (7.4.24)\\ \varphi_i &= \tan(\theta) f_i \ , & (7.4.25)\\ A^i &= \frac{-1}{4\mathcal{V}_f \cot^{-7/3}\theta} G^i u + b^i \ , & (7.4.26)\end{aligned}$$

where the new transverse coordinate u is related to z by

$$\partial_u = (\mathcal{V}_f^{1/2}\cot^{1/2}\theta\,\mathrm{cosec}\,\theta)\partial_z \ . \qquad (7.4.27)$$

Here, C and b^i are constants of integration. Recall, that once the Kähler potential is explicitly given, the volume $\mathcal{V}$ is known, via (7.4.10), as a function of the moduli φ^i. By $\mathcal{V}_f$ we mean this function but with the fields φ^i replaced by f^i. Since the volume is a homogeneous function of degree 7/3 in its arguments this means $\mathcal{V}$ and $\mathcal{V}_f$ are related by

$$\mathcal{V}_f = \mathcal{V}\cot^{7/3}\theta \ . \qquad (7.4.28)$$

The angle θ is fixed by

$$\cos\theta\cot^6\theta = \left(\frac{\mathcal{V}_f}{\mathcal{V}_0}\right)^{3/2} \ , \qquad (7.4.29)$$

where $\mathcal{V}_0$ is another constant of integration. Finally, the new fields f_i are linear functions

$$f_i = \frac{1}{4}G_i w + k_i \ , \qquad (7.4.30)$$

in the transverse coordinate w defined by

$$\partial_w = (\mathcal{V}_f^{3/2}\tan^{9/2}\theta\sec\theta)\partial_z \ , \tag{7.4.31}$$

and k_i are further constants of integration. This completes the most general domain wall solution to four-dimensional $\mathcal{N}=1$ supergravity theories from M-theory flux compactifications on G_2 spaces.

Note that the above solutions display some apparent singularities in the cases $\sin\theta = 1$ or $\cos\theta = 1$ which we have previously encountered when matching to membrane and five-brane sources. However, these singularities are not real but can be removed by introducing a new quantity n defined by either one of the two equivalent relations

$$n = \frac{3\ln|\cot\theta|}{7\ln|\cos\theta| - 5\ln|\sin\theta|} \ , \qquad \left(\frac{\mathcal{V}}{\mathcal{V}_0}\right)^n = |\cot\theta| \ . \tag{7.4.32}$$

In the above solution, we can in general eliminate θ in favour of n.

In this new form one can explicitly consider the cases $\sin\theta = 1$ and $\cos\theta = 1$ without encountering any singularities. They lead to the particularly simple solutions

$\cos\theta = 1$	$\sin\theta = 1$
$\partial_w = \mathcal{V}^{9/10}\partial_z$	$\partial_u = \mathcal{V}^{1/2}\partial_z$
$f_i = \mathcal{V}^{3/5}\varphi_i = \frac{1}{4v}G_i w + k_i$	$f_i = \mathcal{V}^{3/7}\varphi_i = \text{const.}$
$A^i = \text{const.}$	$A^i = -\frac{1}{4\mathcal{V}}G^i u + b^i$
$a = a_0 + \ln\mathcal{V}^{3/10}$	$a = a_0 + \ln\mathcal{V}^{3/14}$

(7.4.33)

The results in this section represent the most general supersymmetric domain wall solution to four-dimensional supergravities that arise from compactification of M-theory on G_2 manifolds with flux. In particular, the solution with $\cos\theta = 1$ in the above table is the appropriate one to match to a wrapped fivebrane source while the solution for $\sin\theta = 1$ can be matched to a membrane source. We note the linear dependence of the moduli fields on the natural transverse coordinate w or u, which will have the effect of causing fields to diverge at large distances. In particular, the solutions do not approach four-dimensional Minkowski space asymptotically. This behaviour is typical for supergravity domain walls supported by potentials without a minimum at any finite field value [135]. We will discuss curvature singularities in section 7.4.5 below.

7.4.4 Comparison with eleven dimensions

As a check of our results, we shall now test for compatibility between the four- and eleven-dimensional relations derived from the Killing spinor equations. Our strategy will be to link the four- and eleven-dimensional quantities and then uplift the four-dimensional equations to eleven dimensions, and see if they agree.

We firstly relate the quantities. By comparing the four-dimensional and eleven-dimensional line elements ds_4^2 and ds_{11}^2 we see that

$$\begin{aligned} \partial_y &= \pm\sqrt{\mathcal{V}}\partial_z \, , && (7.4.34) \\ \alpha &= a - \frac{1}{2}\ln(\mathcal{V}) \, . && (7.4.35) \end{aligned}$$

Using the result from chapter 2 it is clear that to first order

$$\partial_y h_{AB} = \frac{1}{2}(P_{27}\partial_y\varphi)_{(A}{}^{CD}\varphi_{B)CD} + \frac{1}{63}(\partial_y\varphi)\lrcorner\varphi g^{(0)}_{AB} \, . \quad (7.4.36)$$

Making use of relations (7.4.8) and (7.4.7), we simply substitute these results into the four-dimensional bosonic equations for appropriate sign choice in (7.4.34) and get the following

$$\begin{aligned} \partial_y\alpha &= \frac{1}{144}G\lrcorner\Phi\cos\theta - \frac{1}{3}K\sin\theta \, , && (7.4.37) \\ \partial_y\mathrm{tr}(h) &= -\frac{5}{72}G\lrcorner\Phi\cos\theta + \frac{7}{3}K\sin\theta \, , && (7.4.38) \\ \partial_y(P_{27}h)_{AB} &= -\frac{1}{6}(P_{27}G)_{(A}{}^{CDE}\Phi_{B)CDE}\cos\theta \, , && (7.4.39) \\ \partial_y\theta &= -\frac{1}{48}G\lrcorner\Phi\sin\theta - K\cos\theta \, , && (7.4.40) \\ G\lrcorner\Phi &= 24K\cos\theta \, , && (7.4.41) \\ J\lrcorner\varphi &= 21K\cos\theta + \frac{5}{8}G\lrcorner\Phi\sin\theta \, . && (7.4.42) \end{aligned}$$

Although the equations linking components of the flux are superficially different from the eleven-dimensional ones, they are easily shown to be equivalent.

We note here that while the four-dimensional equations did not involve linearision in the flux while the eleven-dimensional ones did. Nevertheless, it turns out by comparing (7.4.37)–(7.4.42) with (7.2.17)–(7.2.21) that the two sets of equations are equivalent at the zero mode level.

7.4.5 Curvature singularities

Another feature of the four-dimensional equations that we should compare with eleven-dimensions is the variation of the volume as a function of the transverse coordinate. A zero of this volume at any particular point in the transverse space implies, of course, a curvature singularity of the internal space. However, it can be shown that such a vanishing internal volume also leads to a four-dimensional curvature singularity.

The z-variation of the volume is described by (7.4.23) which uplifts to give the following

$$\partial_y \mathcal{V} = \left[-\frac{5}{144} G \lrcorner \Phi \cos\theta + \frac{7}{6} K \sin\theta \right] \mathcal{V} \, . \tag{7.4.43}$$

This is the generalisation to all orders in flux of the eleven-dimensional result (7.2.36) in the zero-mode regime. When we consider the second derivative of (7.4.43), using our four-dimensional equations, we get

$$\partial_y^2 \mathcal{V} = \frac{-1}{12\mathcal{V}} \left[(G_i \varphi^i)^2 (2\sin^2\theta - 5) + \frac{1}{2} G_i G^i (2\sin^2\theta + 5) \right] \, , \tag{7.4.44}$$

which is quadratic in the flux. Writing this in eleven-dimensional language gives

$$\partial_y^2 \mathcal{V} = \frac{-1}{12\mathcal{V}} (2\sin^2\theta - 5) \left[\int_{M_7} \varphi \wedge G \right]^2 - \frac{1}{6} (2\sin^2\theta + 5) \int_{M_7} \star G \wedge G \, . \tag{7.4.45}$$

This vanishes at linear order in flux which is consistent with our earlier findings in (7.2.37). These implied a linear behaviour of the volume and, therefore, its vanishing at some finite coordinate y. However, the above result, good to all orders in flux, shows that, in fact, the volume varies in a more complicated way. In particular, the vanishing of the volume at some finite y cannot be deduced generically at this stage. To decide whether or not the volume vanishes one may study specific examples of G_2 manifolds where $\mathcal{V}$ is given as an explicit function of the fields, as in [66].

Chapter 8

(massive) IIA on $SU(3)$-structure manifolds

The outline of this chapter is as follows. In section 8.1 we perform a Kaluza-Klein reduction of massive Type IIA supergravity to an $\mathcal{N}=2$ effective four-dimensional theory by deriving the gravitino mass matrix. In section 8.2 we will go on to derive the superpotential and Kähler potential of the resulting $\mathcal{N}=1$ effective theory and show how the $\mathcal{N}=2$ multiplets break to $\mathcal{N}=1$ superfields. In section 8.3 we will go through an explicit example of such a compactification on the $SU(3)$-structure manifold $SU(3)/U(1)\times U(1)$. We will derive the effective theory for compactification on this coset and find an explicit supersymmetric minimum where all the fields are stabilised at non-trivial vacuum expectation values.

8.1 Reduction of the IIA action

In this section we will consider reducing the ten-dimensional action for massive Type IIA supergravity on a general manifold with $SU(3)$ structure. We will begin by summarising Romans' massive Type IIA supergravity. We will then show how to decompose the ten-dimensional metric, Ricci scalar, dilaton, form fields and gravitino. Reducing the terms that give gravitino mass terms will lead to an effective

$\mathcal{N}=2$ theory, which will be specified by the four dimensional gravitino mass matrix.

8.1.1 Action and field content

The action for massive Type IIA supergravity, first outlined in [136], in the Einstein frame reads

$$\begin{aligned}
S^{10}_{IIA} &= \int\Bigg(\frac{1}{2}\hat{R}\star 1-\frac{1}{4}d\hat{\phi}\wedge\star d\hat{\phi}-\frac{1}{4}e^{-\hat{\phi}}\hat{F}_3\wedge\star\hat{F}_3-\frac{1}{4}e^{\frac{1}{2}\hat{\phi}}\hat{F}_4\wedge\star\hat{F}_4 \\
&- m^2e^{\frac{3}{2}\hat{\phi}}\hat{B}_2\wedge\star\hat{B}_2-m^2e^{\frac{5}{2}\hat{\phi}}\star 1 \\
&+ \frac{1}{4}d\hat{C}_3\wedge d\hat{C}_3\wedge\hat{B}_2+\frac{1}{6}md\hat{C}_3\wedge\hat{B}_2\wedge\hat{B}_2\wedge\hat{B}_2+\frac{1}{20}m^2\hat{B}_2\wedge\hat{B}_2\wedge\hat{B}_2\wedge\hat{B}_2\wedge\hat{B}_2\Bigg) \\
&+ \int\sqrt{-\hat{g}}d^{10}X\Bigg[-\hat{\bar{\Psi}}_M\Gamma^{MNP}D_N\hat{\Psi}_P-\frac{1}{2}\hat{\bar{\lambda}}\Gamma^M D_M\hat{\lambda}-\frac{1}{2}(d\hat{\phi})_N\hat{\bar{\lambda}}\Gamma^M\Gamma^N\hat{\Psi}_M \\
&- \frac{1}{96}e^{\frac{1}{4}\hat{\phi}}(\hat{F}_4)_{PRST}\left(\hat{\bar{\Psi}}^M\Gamma_{[M}\Gamma^{PRST}\Gamma_{N]}\hat{\Psi}^N+\frac{1}{2}\hat{\bar{\lambda}}\Gamma^M\Gamma^{PRST}\hat{\Psi}_M+\frac{3}{8}\hat{\bar{\lambda}}\Gamma^{PRST}\hat{\lambda}\right) \\
&+ \frac{1}{24}e^{-\frac{1}{2}\hat{\phi}}(\hat{F}_3)_{PRS}\left(\hat{\bar{\Psi}}^M\Gamma_{[M}\Gamma^{PRS}\Gamma_{N]}\Gamma_{11}\hat{\Psi}^N+\hat{\bar{\lambda}}\Gamma^M\Gamma^{PRS}\Gamma_{11}\hat{\Psi}_M\right) \\
&+ \frac{1}{4}me^{\frac{3}{4}\hat{\phi}}\hat{B}_{PR}\left(\hat{\bar{\Psi}}^M\Gamma_{[M}\Gamma^{PR}\Gamma_{N]}\Gamma_{11}\hat{\Psi}^N+\frac{3}{4}\hat{\bar{\lambda}}\Gamma^M\Gamma^{PR}\Gamma_{11}\hat{\Psi}_M+\frac{5}{8}\hat{\bar{\lambda}}\Gamma^{PR}\Gamma_{11}\hat{\lambda}\right) \\
&- \frac{1}{2}me^{\frac{5}{4}\hat{\phi}}\hat{\bar{\Psi}}_M\Gamma^{MN}\hat{\Psi}_N-\frac{5}{4}me^{\frac{5}{4}\hat{\phi}}\hat{\bar{\lambda}}\Gamma^M\hat{\Psi}_M+\frac{21}{16}me^{\frac{5}{4}\hat{\phi}}\hat{\bar{\lambda}}\hat{\lambda}\Bigg]\ . \qquad (8.1.1)
\end{aligned}$$

This action is a generalisation of the Type IIA supergravity that is obtained from the low-energy limit of Type IIA string theory, although some care must be taken when taking the massless limit $m\to 0$ [136].

We now turn to notation and field content. The indices $M,N\ldots$ run from 0 to 9, and the ten dimensional space-time coordinates are X^M. In the Neveau-Schwarz-Neveau-Schwarz (NS-NS) sector the action contains the bosonic fields $\hat{\phi},\hat{B}_2,\hat{g}$, which are the ten-dimensional dilaton, a massive two-form and the metric, together with the fermionic fields $\hat{\Psi},\hat{\lambda}$, which are the gravitino and dilatino. The Ramond-Ramond (RR) sector contains the three-form $\hat{C}_3$ and a one-form $\hat{A}^0$ which is eliminated by a gauge transformation of $\hat{B}_2$ as in [136]. The field strengths in the action are given by

$$\hat{F}_4 := d\hat{C}_3 + m\hat{B}_2 \wedge \hat{B}_2 \,, \tag{8.1.2}$$

$$\hat{F}_3 := d\hat{B}_2 \,. \tag{8.1.3}$$

Note that, in contrast to the massless case, $\hat{F}_4$ will not in general be closed, and that due to the equations of motion neither field strength will in general be co-closed.

8.1.2 Decomposing the metric

We now consider reducing the ten dimensional action on a manifold endowed with $SU(3)$ structure. We split the ten-dimensional space-time coordinates as $(X^M) = (x^\mu, y^n)$ with external indices $\mu, \nu \ldots = 0, 1, 2, 3$ and internal indices $m, n \ldots = 4 \ldots 9$. Reflecting the fact that we want the internal space to be compact with compactification radii significantly smaller than any length scales we wish to consider in four dimensions, we decompose the ten-dimensional metric into a sum of four-dimensional and six-dimensional metrics

$$\hat{g}_{MN}(X)dX^M dX^N = \Delta(y) g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(x, y) dy^m dy^n. \tag{8.1.4}$$

$\Delta(y)$ is a possible warp factor which will give the four dimensional metric dependence on the internal coordinates. In section (8.2.1) we will discuss the most general solution of massive Type IIA supergravity on manifolds with $SU(3)$ structure [46] that preserves $\mathcal{N} = 1$ supersymmetry, where it was shown that $\Delta(y)$ is in fact constant. We therefore consistently set it to unity. We note, however, that in the case where supersymmetry is completely broken this warp factor may be non vanishing. (8.1.4) also determines how the Dirac matrices decompose and so we have

$$\Gamma^\mu := \gamma^\mu \otimes \gamma_7 \,, \qquad \Gamma^m := \gamma \otimes \gamma^m \,, \tag{8.1.5}$$

where $\{\gamma^\mu\}$, $\{\gamma^m\}$ furnish representations of the four- and six-dimensional Dirac matrices respectively.

8.1.3 Ricci scalar reduction

Given the choice of metric ansatz above, the ten-dimensional Ricci scalar can be written as

$$\hat{R} = R + R_6 - g^{mn}\nabla^2 g_{mn} - \frac{1}{4}g^{mn}g^{pq}\left(\partial g_{mn}\cdot\partial g_{pq} - 3\partial g_{mp}\cdot\partial g_{nq}\right) \ , \tag{8.1.6}$$

where R, R_6 are the four- and six-dimensional Ricci scalars respectively. ∂, ∇ are four-dimensional derivatives, with $\cdot$ representing contraction over four-dimensional indices. We shall reduce both the Einstein-Hilbert and dilaton kinetic terms at the same time, which are given from (8.1.1) as

$$S^{10}_{EH,D} = \int d^{10}x\sqrt{-g}\left(\frac{1}{2}\hat{R} - \frac{1}{4}\partial_M\hat{\phi}\partial^M\hat{\phi}\right) \ . \tag{8.1.7}$$

Following the reduction of these terms, there are three field redefinitions for the effective four-dimensional action that are needed to put the kinetic terms in canonical form

$$g_{\mu\nu} \ \rightarrow \ \mathcal{V}^{-1}g_{\mu\nu} \ , \tag{8.1.8}$$

$$g_{mn} \ \rightarrow \ e^{-\hat{\phi}/2}g_{mn} \ , \tag{8.1.9}$$

$$\phi \ := \ \hat{\phi} - \frac{1}{2}\ln\mathcal{V} \ , \tag{8.1.10}$$

where ϕ is the four-dimensional dilaton. This gives a final form for the four-dimensional action of

$$S^4_{EH,D} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}e^{3\phi/2}\mathcal{V}^{-1/4}R_6 - \partial\phi\cdot\partial\phi + \frac{1}{8\mathcal{V}}\int d^6x\sqrt{g}\partial g_{mn}\cdot\partial g^{mn}\right) \ . \tag{8.1.11}$$

Our task is then to evaluate the internal integral in terms of the $SU(3)$-structure forms, which is possible via the induced metric, as discussed in section 2.2.5. This allows us to write

$$\frac{1}{2}\partial g_{mn}\cdot\partial g^{mn} = -\partial J_{mn}\cdot\partial J^{mn} + \frac{1}{8}\partial\Omega_{mnp}\cdot\partial\overline{\Omega}^{mnp} \ , \tag{8.1.12}$$

which we will use later in finding the Kähler potential.

8.1.4 The Kaluza-Klein expansion forms

It was suggested in [73], and later developed in [137, 138] that a suitable basis for Kaluza-Klein reduction on manifolds of $SU(3)$ structure is given by two-forms ω_i, three-forms α_A, β^A and four-forms $\widetilde{\omega}^i$ obeying the algebraic relations

$$\begin{aligned} \int \omega_i \wedge \widetilde{\omega}^j = \delta_i^j \ , \qquad & \int \alpha_A \wedge \beta^B = \delta_A^B \ , \\ \int \alpha_A \wedge \alpha_B \ = \ & \int \beta^A \wedge \beta^B = 0 \end{aligned} \tag{8.1.13}$$

and the differential relations

$$\begin{aligned} d\omega_i &= E_{iA}\beta^A - F_i^A \alpha_A \ , \\ d\alpha^A &= E_{iA}\widetilde{\omega}^i \ , \\ d\beta_A &= F_i^A \widetilde{\omega}^i \ , \\ d\widetilde{\omega}^i &= 0 \ , \end{aligned} \tag{8.1.14}$$

where the matrices E_{iA} and F_i^A are constant. In the limit where $E_{iA}, F_i^A \to 0$, we recover the usual set of harmonic forms for a Calabi-Yau compactification: $\{\omega_i, \widetilde{\omega}^j\}_{i,j=1}^{h^{1,1}}$, α_0, β^0, $\{\alpha_a, \beta^b\}_{a,b=1}^{h^{2,1}}$, where the $h^{p,q}$s are the Hodge numbers of the manifold. For the case where $E_{iA}, F_i^A \neq 0$, however, it has been shown in [138] that the relevant forms do not carry topological information, and so there is no metric-independent interpretation of the expansion forms.

Forms satisfying (8.1.13) and (8.1.14) were shown to be the correct basis for the case of half-flat manifolds with Calabi-Yau mirror manifolds. It is natural to extend their use to general half-flat manifolds, and it has been conjectured that such forms could in fact be applied to general $SU(3)$-structure compactifications [138]. With this understood, we shall proceed to make use of them whilst bearing in mind that other bases for Kaluza-Klein reduction are not mathematically excluded.

8.1.5 Decomposing the form fields and fluxes

We decompose the ten-dimensional form fields in the following way:

$$\hat{B}_2(X) = B(x) + \mathring{B}(y) + b(x,y) \ , \tag{8.1.15}$$

$$\hat{C}_3(X) = C(x) + \mathring{C}(y) + c(x,y) \ . \tag{8.1.16}$$

Here B and C are external two and three-forms respectively. $\mathring{B}$ and $\mathring{C}$ are internal two and three-forms with no dependence on external co-ordinates. They give rise to NS-NS and RR flux respectively. b and c are two and three-forms that depend on both the internal and external manifolds. Using the basis (8.1.13) we can expand them as

$$\begin{aligned} b(x,y) &= b^i(x)\omega_i(y) \ , &(8.1.17)\\ c(x,y) &= \xi^A(x)\alpha_A(y) - \widetilde{\xi}_A(x)\beta^A(y) + A^i(x)\wedge\omega_i(y) \ , &(8.1.18)\end{aligned}$$

where A^i are space-time vectors. Given our decomposition of the form fields, the field strengths introduced in (8.1.3) can be written as

$$\begin{aligned} \hat{F}_4 &:= d\hat{C}_3 + m\hat{B}_2\wedge\hat{B}_2 \\ &= d_4(C+c) + d_6(\mathring{C}+c) + m(B+\mathring{B}+b)\wedge(B+\mathring{B}+b) \ , \\ \hat{F}_3 &:= d\hat{B}_2 \\ &= d_4(B+b) + d_6(\mathring{B}+b) \ , \end{aligned} \tag{8.1.19}$$

where d_4 and d_6 denote exterior derivatives on the external and internal spaces respectively. We shall usually suppress these subscripts. Of particular interest are the internal parts of the field strengths (fluxes) which are given by

$$F_3 := d(\mathring{B}+b) =: dB_2 \ , \tag{8.1.20}$$

$$F_4 := d(\mathring{C}+c) + m(\mathring{B}+b)\wedge(\mathring{B}+b) \ . \tag{8.1.21}$$

In contrast to the usual situation for flux compactifications where fluxes obtain their values entirely from the 'background' field strengths $\mathring{B}$ and $\mathring{C}$ (whose precise form is typically not known) the fluxes here can receive contributions from the vacuum expectation values (vevs) of the fields b and c.[1] This difference comes partly because the exterior derivative does not automatically vanish on these fields and partly because the flux F_4 has a non-exact contribution from the second term in (8.1.21). To distinguish between those fluxes that arise in the traditional way and those that arise from vevs, we further define

$$H_3 := d\mathring{B} \; , \qquad G_4 := d\mathring{C} + m\mathring{B} \wedge \mathring{B} \; . \tag{8.1.22}$$

Now, to preserve Poincaré invariance of the four-dimensional theory, all external components of the field strengths must be proportional to the four-dimensional volume form. This restricts us to the only allowed external field strength of

$$(\hat{F}_4)_{\mu\nu\rho\sigma} = f\epsilon_{\mu\nu\rho\sigma} \; , \tag{8.1.23}$$

where, due to its similarity with a similar parameter in the eleven-dimensional case, we will call f a Freud-Rubin parameter. The Freud-Rubin parameter can be calculated in terms of the matter fields by considering the dualisation of the external three-form $C(x)$, which proceeds in a similar manner to chapter 5. Reducing the relevant terms in (8.1.1) gives the four-dimensional action for $C(x)$

$$S_C^{(4)} = \int_{X_4} \left[-\frac{1}{4}\mathcal{V}e^{\frac{1}{2}\hat{\phi}}(dC + mB \wedge B) \wedge \star(dC + mB \wedge B) + \frac{1}{2}AdC \right] \; , \tag{8.1.24}$$

where

$$\begin{aligned} A \;\; := \;\; & \int_Y \Big[d\mathring{C} \wedge \mathring{B} + b \wedge d\mathring{C} + dc \wedge \mathring{B} + \frac{1}{3}m\mathring{B} \wedge \mathring{B} \wedge \mathring{B} + m\mathring{B} \wedge \mathring{B} \wedge b \\ & + m\mathring{B} \wedge b \wedge b + \frac{1}{3}mb \wedge b \wedge b \Big] \; . \end{aligned} \tag{8.1.25}$$

[1] We note here that, as can be seen from (8.1.20) and (8.1.21), the splitting of the two types of contribution to the flux is arbitrary. We could have defined $\mathring{B}$ and $\mathring{C}$ to include the vevs of the scalars and then the scalars would have zero vevs by definition. We have, however, chosen to keep the distinction between the two types more apparent by considering $\mathring{B}$ and $\mathring{C}$ as arising only from sources other than the scalar vevs.

To dualise C we follow the discussion in [139] and add a Lagrange multiplier λ

$$S_C^{(4)} = \int_X \left[-\frac{1}{4}\mathcal{V}e^{\frac{1}{2}\hat{\phi}}(dC + mB \wedge B) \wedge \star(dC + mB \wedge B) + \frac{1}{2}AdC + \frac{1}{2}\lambda dC \right] . \tag{8.1.26}$$

Taking the equation of motion for C and substituting in (8.1.23) then gives

$$\star(dC + mB \wedge B) = \mathcal{V}^{-1}e^{-\frac{1}{2}\hat{\phi}}\,(A + \lambda) = -f \;, \tag{8.1.27}$$

which allows us to write f in terms of the four-dimensional constant λ and the integral (8.1.25). We note here that we do not need to perform a similar dualisation for $\hat{B}_2$, since although a massless two-form would have been dual to a scalar, a massive two-form will be dual to a massive vector, which we are not considering in our analysis.

8.1.6 The geometrical moduli

We now turn to the fields arising from metric deformations. From general $\mathcal{N} = 2$ supergravity considerations in chapter 4 and [107, 140], we know that the complex structure deformations z^a span a special Kähler manifold $\mathcal{M}^{cs}$ with a unique holomorphic three-form Ω^{cs}, which has *periods* Z^A and $F_A(Z^A)$, that are homogeneous functions of the z^as. The Kähler potential is then given by the symplectic inner product

$$K^{cs} := -\ln i \left\langle \Omega^{cs}, \overline{\Omega}^{cs} \right\rangle = -\ln i \left[\bar{Z}^A F_A - Z^A \bar{F}_A \right] =: -\ln(||\Omega^{cs}||^2 \mathcal{V}) \; . \tag{8.1.28}$$

We can then use the forms in section 8.1.4 to expand Ω^{cs} and re-write the Kähler potential as below

$$\begin{aligned} \Omega^{cs} &= Z^A \alpha_A - F_A \beta^A \;, & (8.1.29)\\ K^{cs} &= -\ln i \int \Omega^{cs} \wedge \overline{\Omega}^{cs} \; . & (8.1.30) \end{aligned}$$

We are now interested in relating Ω^{cs} to the holomorphic three-form Ω in (2.2.13). Writing

$$\Omega^{cs} = \frac{1}{\sqrt{8}}||\Omega^{cs}||\Omega \;, \tag{8.1.31}$$

$g_{\mu\nu}, A^0$	gravitational multiplet
$\xi^0, \widetilde{\xi}_0, \phi, B$	tensor multiplet
b^i, v^i, A^i	vector multiplets
$\xi^a, \widetilde{\xi}_a, z^a$	hypermultiplets

Table 8.1: Table showing the $\mathcal{N} = 2$ multiplets in Type IIA theory

we see that inserting (8.1.31) into (8.1.30) and using (2.2.10) we recover (8.1.28). As a check on this process, we note that inserting the relation (8.1.31) into (2.2.24) and going to a local patch where we can write global holomorphic and anti-holomorphic coordinates, the usual relations for metric variations are obtained

$$\delta g_{\alpha\overline{\beta}} = -i\delta J_{\alpha\overline{\beta}} \ , \qquad \delta g_{\alpha\beta} = -\frac{1}{||\Omega^{cs}||^2}\left(\delta\overline{\Omega}^{cs}\right)_{\alpha}{}^{\gamma\delta}(\Omega^{cs})_{\beta\gamma\delta} \ . \tag{8.1.32}$$

The Kähler structure deformations v^i arise in the usual way, after we expand J in the forms from section 8.1.4, which gives

$$J = v^i\omega_i \ , \tag{8.1.33}$$

$$K = -\ln\frac{4}{3}J\wedge J\wedge J \ . \tag{8.1.34}$$

Inserting (8.1.17) into (8.1.1) and (8.1.33) into (8.1.11) we see that the Kähler structure deformations v^i combine with the NS-NS scalars b^i to span a special Kähler manifold $\mathcal{M}^{SK}$ with Kähler potential (8.1.34).

In summary, the geometrical moduli fields combine with the massless modes of the matter fields to form $\mathcal{N} = 2$ multiplets as shown in Table 8.1. The hypermultiplets span a quaternionic manifold $\mathcal{M}^Q$ with a special Kähler submanifold $\mathcal{M}^{cs}$ and the vector multiplets span the special Kähler manifold $\mathcal{M}^K$.

8.1.7 Decomposing the gravitino

Before we write down the mass matrix for the gravitini, we have to choose an appropriate ansatz for the ten-dimensional gravitino. As discussed in section 2.2.3 the internal manifold, which has $SU(3)$ structure, supports a single globally defined, positive-chirality Weyl spinor η_+ and its charge conjugate η_-, which will have negative chirality. From standard arguments, we expect terms involving other spinors on the internal space to lead to four-dimensional masses at the Kaluza-Klein scale, and so they can be ignored. Given $\mathcal{N}=2$ supersymmetry, we further expect the external degrees of freedom for the gravitino to be given by a single Dirac spinor which can be decomposed as two independent Weyl spinors. The most general spinor ansatz for the ten dimensional gravitino that involves these degrees of freedom is then

$$\hat{\Psi}_M = \psi_{M\alpha} \otimes (a^\alpha \eta_+ + b^\alpha \eta_-) + \psi_M^\alpha \otimes (c_\alpha \eta_+ + d_\alpha \eta_-) \ , \tag{8.1.35}$$

where the indices α, β are $SU(2)$ indices, which imply positive chirality of a spinor when lowered and negative chirality when raised. $a^\alpha, b^\alpha, c_\alpha, d_\alpha$ are complex numbers. $\psi_{\mu 1,2}$ are thus four-dimensional gravitini with positive chirality and charge conjugates $\psi_\mu^{1,2}$, while $\psi_{m1,2}$ are four-dimensional spin-1/2 fields with charge conjugates $\psi_m^{1,2}$. Note that, as previously mentioned in chapter 5, in order not have cross terms between the gravitini and the spin-1/2 fields the gravitini need to be redefined with some combination of the spin-1/2 fields. This does not affect the mass of the gravitini, however, and so will not be considered here.

There are two physical constraints that we impose on the ansatz (8.1.35) to restrict it. The first of these is that the ten-dimensional gravitino should be Majorana. This gives the conditions

$$c_{1,2} = -(b^{1,2})^* \ , \qquad d_{1,2} = -(a^{1,2})^* \ . \tag{8.1.36}$$

The second constraint is that the gravitino ansatz should yield canonical kinetic terms when reduced, which in this case look like

$$S^4_{\text{k.t.}} = -\int \sqrt{-g} d^4x \left(\overline{\psi^1_\mu} \gamma^{\mu\rho\nu} D_\rho \psi_{\nu 2} + \overline{\psi^2_\mu} \gamma^{\mu\rho\nu} D_\rho \psi_{\nu 2} \right) + \text{c.c.} \ , \tag{8.1.37}$$

where c.c. stands for charge conjugate. The kinetic term for the ten-dimensional gravitino reads

$$S^{10}_{\text{k.t.}} = \int \sqrt{-\hat{g}} d^{10}X \left[-\bar{\hat{\Psi}}_M \Gamma^{MNP} D_N \hat{\Psi}_P \right] \ . \tag{8.1.38}$$

Substituting (8.1.35) into (8.1.38) and performing the Weyl rescaling (8.1.8) we get the result that the four-dimensional gravitino kinetic terms will only take the correct form when

$$(a^\alpha)^*(a^\beta) + (b^\alpha)^*(b^\beta) = \frac{1}{2}\mathcal{V}^{-1/2}\delta^{\alpha\beta} \ . \tag{8.1.39}$$

Imposing (8.1.36) and (8.1.39), together with the absorption of a constant phase into one of the spinor degrees of freedom, gives the following form for the gravitino ansatz

$$\begin{aligned}\hat{\Psi}_M = \frac{1}{2}\mathcal{V}^{-1/4} \quad \Big[\quad & \psi_{M1} \otimes \left(\sqrt{1/2+\varepsilon}\,\eta_+ + \sqrt{1/2-\varepsilon}\,e^{i\theta}\eta_-\right) \\ + \quad & \psi_{M2} \otimes \left(\sqrt{1/2-\varepsilon}\,\eta_+ - \sqrt{1/2+\varepsilon}\,e^{i\theta}\eta_-\right)\Big] + \text{c.c.}\end{aligned} \tag{8.1.40}$$

ε can be chosen at convenience by making a further spinor redefinition, while θ is a phase that is not fixed by physical considerations and cannot be absorbed into a spinor redefinition.

Rather than leave these remaining parameters in, we note that upon performing the reduction of terms that give a gravitino mass, it is most convenient to choose $\varepsilon = 0$, while θ can be eliminated by making the redefinitions below, which will not affect the four-dimensional physics

$$\Omega \to e^{i\theta}\Omega \ , \qquad M_{3/2} \to e^{i\theta}M_{3/2} \ , \tag{8.1.41}$$

where $M_{3/2}$ is a gravitino mass. This gives us the working ansatz for the gravitino

$$\hat{\Psi}_M = \frac{1}{2\sqrt{2}}\mathcal{V}^{-1/4}\left[\psi_{M1}\otimes(\eta_+ + \eta_-) + \psi_{M2}\otimes(\eta_+ - \eta_-)\right] + \text{c.c.} \tag{8.1.42}$$

8.1.8 Gravitino mass matrix

We are interested in the gravitino mass matrix of the $\mathcal{N}=2$ four-dimensional theory. The terms in the ten-dimensional action (8.1.1) which will contribute to the gravitino masses are

$$
\begin{aligned}
S^{10}_{\text{mass}} = \int \sqrt{-\hat{g}} d^{10}X \Big[& - \hat{\overline{\Psi}}_\mu \Gamma^{\mu n \nu} D_n \hat{\Psi}_\nu \\
& - \frac{1}{96} e^{\frac{1}{4}\hat{\phi}} (\hat{F}_4)_{prst} \hat{\overline{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{prst} \Gamma_{\nu]} \hat{\Psi}^\nu \\
& - \frac{1}{96} e^{\frac{1}{4}\hat{\phi}} (\hat{F}_4)_{\rho\sigma\delta\epsilon} \hat{\overline{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{\rho\sigma\delta\epsilon} \Gamma_{\nu]} \hat{\Psi}^\nu \\
& + \frac{1}{24} e^{-\frac{1}{2}\hat{\phi}} (\hat{F}_3)_{prs} \hat{\overline{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{prs} \Gamma_{\nu]} \Gamma_{11} \hat{\Psi}^\nu \\
& + \frac{1}{4} m e^{\frac{3}{4}\hat{\phi}} \hat{B}_{pr} \hat{\overline{\Psi}}^\mu \Gamma_{[\mu} \Gamma^{pr} \Gamma_{\nu]} \Gamma_{11} \hat{\Psi}^\nu \\
& - \frac{1}{2} m e^{\frac{5}{4}\hat{\phi}} \hat{\overline{\Psi}}_\mu \Gamma^{\mu\nu} \hat{\Psi}_\nu \Big] \ .
\end{aligned} \tag{8.1.43}
$$

Using the ansatz (8.1.42), the definitions of J and Ω and the relations in section 2.2.3 as well as the discussion in section 8.1.5 we can derive the resulting four-dimensional masses. After performing the appropriate rescalings as in section 8.1.3 the mass terms can be written as

$$
S^4_{\text{mass}} = \int \sqrt{-g} d^4x \left[S^\alpha{}_\beta \overline{\psi_{\mu\alpha}} \gamma^{\mu\nu} \psi^\beta_\nu + (S_\alpha{}^\beta)^* \overline{\psi^\alpha_\mu} \gamma^{\mu\nu} \psi_{\nu\beta} \right] \ , \tag{8.1.44}
$$

where $\alpha, \beta = 1, 2$ label the gravitini. The mass matrix S is given by

$$
S = \begin{pmatrix} M_1 & D \\ D & M_2 \end{pmatrix} \tag{8.1.45}
$$

with terms defined as below

$$
\begin{aligned}
M_1 &:= \frac{-i}{8} e^{2\phi} \mathcal{V}^{-\frac{1}{2}} \Bigg[\lambda + \int_Y \left(d\mathring{C} \wedge \mathring{B} + \frac{1}{3} m \mathring{B} \wedge \mathring{B} \wedge \mathring{B} \right) + \int_Y (dT + H_3) \wedge U \\
& \quad + \int_Y \left(\frac{1}{3} m T \wedge T \wedge T + m \mathring{B} \wedge T \wedge T + G_4 \wedge T \right) \Bigg] \ , \\
M_2 &:= -M_1|_{U \to \overline{U}} \ , \\
D &:= \frac{-i}{8} e^{2\phi} \mathcal{V}^{-\frac{1}{2}} \int_Y (dT + H_3) \wedge (i \mathcal{V}^{-\frac{1}{2}} e^{-\phi} \Omega^+) \ , \\
T &:= b - iJ \ , \\
U &:= c + i \mathcal{V}^{-\frac{1}{2}} e^{-\phi} \Omega^- = c + i\sqrt{8} \mathcal{V}^{-\frac{1}{2}} ||\Omega^{cs}||^{-1} e^{-\phi} \Omega^{cs-} \ ,
\end{aligned} \tag{8.1.46}
$$

where Ω^+ and Ω^- are the real and imaginary parts of Ω respectively. The four dimensional effective theory will be an $\mathcal{N} = 2$ gauged supergravity. Taking the general form for a gauged supergravity found in [107] we see that using their conventions the gravitino mass matrix is given by

$$S_{\alpha\beta} = \frac{i}{2} P^x_A \sigma^x_{\alpha\beta} L^A \ , \tag{8.1.47}$$

where the P^x_A are prepotentials, $\sigma^x_{\alpha\beta}$ are Pauli matrices and $L^A := e^{\frac{1}{2}K^{cs}} Z^A$, where K^{cs}, Z^A are defined in section 8.1.6. Comparing (8.1.47) with (8.1.46) we can completely determine the Kähler potential of the vector multiplet sector and the prepotentials of the hypermultiplet sector. We will not go on to do this because in the next section we will see that quite generally this theory will not preserve $\mathcal{N} = 2$ supersymmetry in the vacuum and we will instead have to consider specifying an $\mathcal{N} = 1$ effective theory.

8.2 Breaking to $\mathcal{N} = 1$

In this section we will explore the implications of the form of the gravitino mass matrix found in the previous section. In order to do this we will specialise to the case where the internal manifold is a particular class of half-flat manifolds. To motivate this choice we will review the most general supergravity solution of massive Type IIA on manifolds with $SU(3)$ structure that preserves some supersymmetry constructed in [46]. We will then go on to show that for that class of manifold the low energy theory will not preserve $\mathcal{N} = 2$ supersymmetry in the vacuum and in fact will exhibit spontaneous partial supersymmetry breaking to $\mathcal{N} = 1$. In section 8.2.3 we will derive the effective action of the resulting $\mathcal{N} = 1$ theory.

8.2.1 Ten-dimensional massive IIA solutions

In general, the reduction of Type-II supergravities on spaces of $SU(3)$ structure should yield an $\mathcal{N} = 2$ supergravity. There are, however, solutions to (massive) IIA supergravity on manifolds of $SU(3)$ structure that preserve supercharges consistent with $\mathcal{N} = 1$ supersymmetry in four dimensions. These were first considered in

[35, 39, 45], and later generalised in [46]; we shall therefore refer to them as BCLT (Behrndt-Cvetic-Lust-Tsimpis) solutions. We present here a brief summary of the more general solution in ten-dimensional language.

The metric takes the form of (8.1.4), with Δ constant, while the fluxes and form fields for the solution take the values

$$\begin{aligned} m\hat{B}_2 &= \frac{1}{18} f e^{-\hat{\phi}/2} J + m\widetilde{B} \, , \\ \hat{F}_3 &= \frac{4}{5} m e^{7\hat{\phi}/4} \Omega^+ \, , \\ \hat{F}_4 &= f \star \mathbb{1}_4 + \frac{3}{5} m e^{\hat{\phi}} J \wedge J \, , \end{aligned} \tag{8.2.1}$$

where f and $\hat{\phi}$ are constant. $\widetilde{B}$ encodes the non-singlet part of $\hat{B}_2$ and so obeys $\widetilde{B} \wedge J \wedge J = 0$, but is otherwise quite general. A key feature of the solution is that all torsion classes of the compact space vanish except for

$$\begin{aligned} \mathcal{W}_1 &= -i \frac{4}{9} f e^{\hat{\phi}/4} \, , \\ \mathcal{W}_2 &= -2ime^{3\hat{\phi}/4} \widetilde{B} \, . \end{aligned} \tag{8.2.2}$$

Manifolds specified by the torsion classes (8.2.2) are half-flat, and will play an important role in upcoming sections where we will restrict the internal manifold to lie in this class. We note here that we can always use this type of 'internal' information from a solution in constructing four-dimensional effective actions.

It is informative to see how the fluxes arise in this solution. Considering the torsion classes (8.2.2) and the relation (8.1.14) and comparing the fluxes (8.1.20) and (8.1.21) with (8.2.1), we see that the solution corresponds precisely to the case where the fluxes arise purely from the scalar vevs. This will be an important observation later on when we consider what types of fluxes break supersymmetry. A further result of the solution that we shall make use of is that

$$M_{3/2} = \Delta \left(\frac{\alpha}{|\alpha|} \right)^{-2} \left[-\frac{1}{5} m e^{5\hat{\phi}/4} + \frac{i}{6} f e^{\hat{\phi}/4} \right] \, , \tag{8.2.3}$$

where $M_{3/2}$ is the value of the four-dimensional gravitino mass for this solution and α is a constant related to the spinor phase θ that we discussed in section 8.1.7 and can be consistently set to unity.

8.2.2 Spontaneous partial supersymmetry breaking

We now want to consider the case where the D terms in the mass matrix vanish. From (8.1.46) and (8.2.2) we see that for half flat manifolds $d\Omega^+ = 0$ and so the D terms indeed vanish. The mass matrix diagonalises under this constraint and we see that there appears a mass gap ΔM^2 between the two gravitini given by

$$\begin{aligned}\Delta M^2 &= |M_2|^2 - |M_1|^2 \,, \\ &= \frac{1}{32} e^{3\phi} \mathcal{V}^{-1} \Bigg[\int_Y F_3 \wedge \Omega^- \int_Y \left(\frac{1}{3} m J \wedge J \wedge J + F_4 \wedge J \right) \\ &\quad + \int_Y dJ \wedge \Omega^- \int_Y \left(\frac{1}{6} f e^{\hat{\phi}/2} J \wedge J \wedge J + m B_2 \wedge J \wedge J \right) \Bigg] \,. \qquad (8.2.4)\end{aligned}$$

It is interesting to consider how this mass gap depends on the fluxes. In massless Type IIA supergravity such a mass gap requires both RR and NS-NS fluxes to be non-vanishing [141] (despite the usual subtleties in doing so, is it possible to see this by taking the limits $dJ, m \rightarrow 0$ in (8.2.4) above). We see that this is not the case here. Either type of flux by itself will generate a mass gap due to a non-vanishing Freud-Rubin parameter [2]. Hence, given general fluxes, the masses of the gravitini are non-degenerate. This implies that we no longer have $\mathcal{N} = 2$ supersymmetry. Indeed such a mass gap corresponds to partial supersymmetry breaking with $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ for a physically massless lighter gravitino or full supersymmetry breaking with $\mathcal{N} = 2 \rightarrow \mathcal{N} = 0$ for a physically massive lighter gravitino

In a Minkowski background, physically massless particles simply have zero mass. In anti-de Sitter (AdS) backgrounds, however, physically massless particles can have non-zero masses [142–144]. This is the case here and so although the masses M_1 and M_2 in (8.1.46) are non-zero for non vanishing fluxes one of them may still be physically massless. As we saw in section 8.2.1 fluxes which arise from vevs can preserve $\mathcal{N} = 1$ supersymmetry and therefore have a physically massless gravitino. We can

[2] The case where the Freud-Rubin parameter vanishes will not be a proper supergravity solution and so we do not consider it here.

then check that one of our gravitini is indeed physically massless by substituting the solution described in section 8.2.1 into our mass matrix (8.1.46) and checking that one of the gravitini has a mass corresponding to the gravitino mass found in the solution.

Putting the solution (8.2.1) into the gravitino mass matrix and taking care with the rescalings in section 8.1.3, we find firstly that $D = 0$. This means that $\psi_{1,2}$ are both mass eigenstates, with eigenvalues that obey

$$\begin{aligned} M_1 &= \frac{1}{5} m e^{5\hat{\phi}/4} - \frac{i}{6} f e^{\hat{\phi}/4} \, , \\ M_2 &= -3M_1 \, . \end{aligned} \tag{8.2.5}$$

Comparison with (8.2.3) gives that $|M_1| = |M_{3/2}|$. We therefore see that for the BCLT background, a mass gap opens up for the two gravitini such that the ψ^1_μ is physically massless and ψ^2_μ is physically massive. With a slight abuse of terminology we shall therefore refer to the lower mass gravitino as massless and the higher mass one as massive.

For an inexhaustive list of literature discussing partial supersymmetry breaking see [108, 141, 145–149]. Following their discussions we briefly summarise how the matter sector of the theory is affected by the breaking. In the $\mathcal{N} = 2$ theory the fields were grouped into multiplets as described in Table 8.1. Once supersymmetry is broken these multiplets should split up into $\mathcal{N} = 1$ multiplets. The $\mathcal{N} = 2$ gravitational multiplet will need to split into a $\mathcal{N} = 1$ 'massless' gravitational multiplet and a 'massive' spin-$\frac{3}{2}$ multiplet

$$\left(g_{\mu\nu}, \psi_1, \psi_2, A^0\right) \rightarrow \text{massless } \left(g_{\mu\nu}, \psi_1\right) \; + \; \text{massive } \left(\psi_2, A^0, A^1, \phi_1, \phi_2\right) \, . \tag{8.2.6}$$

Here A^1 is a vector field which has to come from one of the vector multiplets and ϕ_1 and ϕ_2 are spin-$\frac{1}{2}$ fermions which come from a hypermultiplet. The N_V $\mathcal{N} = 2$ vector multiplets break into n_v massless $\mathcal{N} = 1$ vector multiplets and n_c massless chiral multiplets (with the other fields forming massive multiplets) such that the scalar components of the chiral multiplets span a Kähler manifold $\mathcal{M}^{KV} \subset \mathcal{M}^{SK}$. The N_H

$\mathcal{N} = 2$ hypermultiplets break into n_h massless $\mathcal{N} = 1$ chiral multiplets and $N_H - n_h$ massive chiral multiplets with $n_h \leq \frac{1}{2}N_H$. The scalar components of the massless chiral multiplets span a Kähler manifold $\mathcal{M}^{KH} \subset \mathcal{M}^Q$. With mass gaps appearing throughout the matter spectrum we can consider working with an effective $\mathcal{N} = 1$ theory by integrating out the higher physical mass modes. For the case of scalars and fermions this amounts to setting them to zero thereby truncating the matter spectrum of the theory. It is not immediately clear from the above considerations exactly which fields to truncate, however we will return to this question in section 8.2.3 when we construct the $\mathcal{N} = 1$ effective theory.

It is interesting to consider the case where $\mathring{B} = 0 = \mathring{C}$ and the flux arises solely from the vevs of the scalar fields. Then any vacuum of the truncated $\mathcal{N} = 1$ theory where the scalars have non-vanishing vevs for which $\Delta M^2 \neq 0$ will indeed be a valid vacuum of the full $\mathcal{N} = 2$ theory. We will use this observation to find such vacua in section 8.3.2.

A further reason for taking $\mathring{B} = 0 = \mathring{C}$ comes from the consideration of section 6.3.4. Since in that section, the flux G decoupled from the low-energy degrees of freedom, we might reasonably expect the fluxes to do the same here, although a full analysis of mass operators in the BCLT background would involve a lengthy calculation. We further believe that the tadpoles discussed in [92] can be avoided by this decoupling.

8.2.3 The $\mathcal{N} = 1$ effective theory

We are interested in constructing the effective $\mathcal{N} = 1$ theory of the physically massless modes. To do this we must explicitly determine how the $\mathcal{N} = 2$ multiplets in Table 8.1 break into $\mathcal{N} = 1$ superfields and which of these superfields are physically massive or massless. The form of (8.1.46) suggests that T and U are the correct variables to expand in the chiral superfields. To prove this is the case we will need to show that these superfields span a Kähler manifold with a Kähler potential which matches the one that will be derived from the gravitino mass.

We now turn to the calculation of the $\mathcal{N}=1$ superpotential and Kähler potential. In the effective $\mathcal{N}=1$ theory the remaining gravitino mass can be written as

$$M_{3/2} = e^{\frac{1}{2}\mathcal{K}} W \ , \tag{8.2.7}$$

where $\mathcal{K}$ is the Kähler potential and W is the superpotential of the theory. It is only this Kähler-invariant combination of W and $\mathcal{K}$ that has any physical significance, although it is still natural to decompose (8.2.7) as

$$\begin{aligned} e^{\frac{1}{2}\mathcal{K}} &= \frac{e^{2\phi}}{\sqrt{8}\mathcal{V}^{\frac{1}{2}}} \ , && (8.2.8) \\ W &= \frac{-i}{\sqrt{8}}\Bigg[\lambda + \int_Y \left(d\mathring{C} \wedge \mathring{B} + \frac{1}{3} m \mathring{B} \wedge \mathring{B} \wedge \mathring{B}\right) && (8.2.9) \\ &\quad + \int_Y \left(\frac{1}{3} m T \wedge T \wedge T + m\mathring{B} \wedge T \wedge T + G \wedge T + (dT + H) \wedge U\right)\Bigg] \ . \end{aligned}$$

This gives a general form for the superpotential and Kähler potential coming from the $\mathcal{N}=1$ effective action following spontaneous breaking of the $\mathcal{N}=2$ theory for massive IIA on manifolds of $SU(3)$ structure. The theory will also have D-terms corresponding to the off-diagonal elements of the $\mathcal{N}=2$ gravitno mass matrix, D in (8.1.46), which vanish for half-flat manifolds. We will now express W and $\mathcal{K}$ in four-dimensional language, assuming that we can expand in the forms of section 8.1.4 so that

$$T = T^i \omega_i \ , \qquad U = U^A \alpha_A - \widetilde{U}_A \beta^A \ . \tag{8.2.10}$$

We can then interpret $T^i, U^A, \widetilde{U}_A$ as the scalar components of chiral superfields, of which the superpotential should be a holomorphic function. Substituting (8.2.10) into (8.2.9), we can write the superpotential as

$$\begin{aligned} W &= \frac{-i}{\sqrt{8}}\Big[\lambda' + G_i T^i + B_{ij} T^i T^j + k_{ijk} T^i T^j T^k + H_A U^A + \widetilde{H}^A \widetilde{U}_A \\ &\quad + (F_i^A \widetilde{U}_A - E_{iA} U^A) T^i\Big] \ , \end{aligned} \tag{8.2.11}$$

where $\lambda', G_i, B_{ij}, k_{ijk}, H_A, \widetilde{H}^A$ are four-dimensional constants given by six-dimensional integrals

$$\begin{array}{rclrcl}\lambda' & = & \lambda + \int_Y (d\mathring{B}\wedge\mathring{C} + \frac{1}{3}m\mathring{B}\wedge\mathring{B}\wedge\mathring{B}) \ , & k_{ijk} & = & \frac{1}{3}m\int_Y \omega_i\wedge\omega_j\wedge\omega_k \ , \\ B_{ij} & = & m\int_Y \mathring{B}\wedge\omega_i\wedge\omega_j \ , & G_i & = & \int_Y G\wedge\omega_i \ , \\ H_A & = & \int_Y H\wedge\alpha_A \ , & \widetilde{H}^A & = & \int_Y H\wedge\beta^A \ .\end{array} \tag{8.2.12}$$

As was discussed in section 8.2.2 turning on fluxes $\mathring{B},\mathring{C}\neq 0$ will, in general, break supersymmetry further. In the case where supersymmetry is completely broken it does not make sense to talk about superpotentials and superfields. If these fluxes are small relative to the flux originating from the scalar vevs, however, then they can be perturbatively included in the superpotentials (8.2.9) and (8.2.11). We therefore display (8.2.11) as an indication of the class of effective theories that can be obtained from the compactification of massive IIA supergravity on spaces of $SU(3)$ structure. These may be of use in, for example, studying $\frac{1}{2}$-BPS states of such theories as in [1].

To be sure of retaining $\mathcal{N}=1$ supersymmetry we will only consider fluxes originating from scalar vevs from now on. In that case the superpotential can be written as

$$W = \frac{-i}{\sqrt{8}}\left[\lambda + \int_Y\left(\frac{1}{3}mT\wedge T\wedge T + dT\wedge U\right)\right] \ . \tag{8.2.13}$$

Having determined the superpotential of the effective theory we can consider the Kähler potential. To prove that (8.2.8) is indeed the correct Kähler potential of the truncated theory we need to explicitly perform the truncation and show that the remaining fields form $\mathcal{N}=1$ superfields, $T^i, U_A, \widetilde{U}_A$, with the corresponding metric. In the Kähler moduli sector it was shown in section 8.1.5 that indeed the scalars b^i and v^i combine into $T^i = b^i - iv^i$ with Kähler potential (8.1.34). In the hypermultiplet sector we have N_H hypermultiplets with $4N_H$ real scalar components which are to be truncated to n_h chiral multiplets with $2n_h$ real components. It seems that the correct superfields to form are then

$$U^A = \xi^A + i\sqrt{8}\mathcal{V}^{-\frac{1}{2}}e^{-\phi}\mathrm{Im}\left(||\Omega^{cs}||^{-1}Z^A\right) \ , \tag{8.2.14}$$

$$\widetilde{U}_A = \widetilde{\xi}_A + i\sqrt{8}\mathcal{V}^{-\frac{1}{2}}e^{-\phi}\mathrm{Im}\left(||\Omega^{cs}||^{-1}F_A\right) \ . \tag{8.2.15}$$

Indeed this form for the superfields has been proposed in [150], and also derived in [151] for the case where the partial supersymmetry breaking is induced through an orientifold projection. In our case, however, things are more simple. The internal manifold is a half-flat manifold which has torsion classes

$$\mathrm{Re}(\mathcal{W}_1) = \mathrm{Re}(\mathcal{W}_2) = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0 \ , \tag{8.2.16}$$

so the general relations for the proposed Kaluza-Klein basis (8.1.14) reduce to

$$\begin{aligned} d\omega_i &= E_i \beta_0 \ , \\ d\alpha_0 &= E_i \widetilde{\omega}^i \ , \\ d\widetilde{\omega}^i &= 0 = d\beta^A = d\alpha_{A\neq 0} \ , \end{aligned} \tag{8.2.17}$$

for $E_i := E_{0i}$. Applying (8.2.17) to (2.2.14) we arrive at

$$dJ = E_i v^i \beta^0 = -\frac{3}{2}\mathrm{Im}\left(\mathcal{W}_1\right)\mathrm{Re}\left(\overline{\Omega}\right) \ , \tag{8.2.18}$$

$$d\Omega = Z^0 E_i \widetilde{\omega}^i = i\mathrm{Im}\left(\mathcal{W}_1\right) J \wedge J + i\mathrm{Im}\left(\mathcal{W}_2\right) \wedge J \ . \tag{8.2.19}$$

Equation (8.2.18) is the motivation behind the statement that the special class of half flat manifolds under consideration do not have any complex structure deformations associated with them. This means that we only have the tensor multiplet and so we only have one chiral superfield left in the truncated theory. This superfield will contain the dilaton ϕ and either ξ^0 or $\widetilde{\xi}_0$. To decide which of the two is to be truncated we can refer to (8.2.19). We see that for our case, $\mathrm{Re}(\Omega) \propto \beta^0$ and $\mathrm{Im}(\Omega) \propto \alpha_0$. And therefore since only the imaginary part of Ω appears in the effective $\mathcal{N} = 1$ theory we should truncate the field associated with β^0, that is $\widetilde{\xi}_0$. Using the restrictions discussed above we can write the remaining superfield as

$$U_0 = \xi^0 + ie^{-\phi}\left(\frac{-4iZ^0}{F_0}\right)^{\frac{1}{2}} \ . \tag{8.2.20}$$

Now inserting (8.1.18) into (8.1.1) we get the kinetic term

$$\begin{aligned} S^U_{kin} = \int \sqrt{-g}d^4x \Bigg[&- \left(\frac{F_0}{-4iZ^0}\right) e^{2\phi}\partial_\mu \left(\xi^0 + ie^{-\phi}\left(\frac{-4iZ^0}{F_0}\right)^{\frac{1}{2}}\right) \\ &\times \partial^\mu \left(\xi^0 - ie^{-\phi}\left(\frac{-4iZ^0}{F_0}\right)^{\frac{1}{2}}\right)\Bigg] \ . \end{aligned} \tag{8.2.21}$$

We see that taking the second derivatives,

$$-\partial_{U^0}\partial_{\bar{U}^0} \ln\left[\frac{e^{4\phi}}{8\mathcal{V}}\right] = \left(\frac{F_0}{-4iZ^0}\right) e^{2\phi} , \tag{8.2.22}$$

and so (8.2.8) is indeed the correct Kähler potential and (8.2.20) is the correct superfield.

8.3 An example: $SU(3)/U(1)\times U(1)$

Having derived in section 8.2.3 the form of the $\mathcal{N}=1$ effective theory on a general manifold with torsion classes (8.2.16), in this section we will look at an explicit example of such a manifold. Denoting the internal manifold by $\mathcal{Y}$ we will consider the coset space

$$\mathcal{Y} = \frac{SU(3)}{U(1)\times U(1)}. \tag{8.3.1}$$

In section 8.3.1 we will derive explicit expressions for J, Ω and the expansion forms on $\mathcal{Y}$. We will then consider the effective theory and derive the superpotential and Kähler potential. Finally we will find supersymmetric minima where all the superfields have non-trivial expectation values.

8.3.1 Geometry of the coset

In general, a coset manifold $\mathcal{Y} := \mathcal{G}/\mathcal{H}$, where $\mathcal{H}\subset\mathcal{G}$, can be given a non-coordinate basis by taking the generators of $\mathcal{G}$ and removing the generators of $\mathcal{H}$ in a way that is consistent with the embedding of $\mathcal{H}$ in $\mathcal{G}$. We can then construct tensor products of this basis, and it turns out that tensors on the coset are heavily restricted by imposing that they remain invariant under the action of any element of $\mathcal{G}$. This restriction allows us to write the most general $\mathcal{G}$-invariant tensors that can exist on the coset.

The particular case $SU(3)/U(1)\times U(1)$ has been considered in [152], where the two $U(1)$ subgroups are naturally identified with the diagonal Gell-Mann matrices.

It was shown that the most general $\mathcal{G}$-invariant two- and three-forms can be written as

$$\begin{aligned} A_{(2)} &= \alpha e^{12} + \beta e^{34} + \gamma e^{56} \, , \\ A_{(3)} &= \delta(e^{136} - e^{145} + e^{235} + e^{246}) + \epsilon(e^{135} + e^{146} - e^{236} + e^{245}) \, , \end{aligned} \quad (8.3.2)$$

where the $\{e^m\}$ form a basis on the coset space, $\alpha \ldots \epsilon$ are complex coefficients and we define $e^{m_1 \ldots m_p} \equiv e^{m_1} \wedge \ldots \wedge e^{m_p}$. Furthermore, by considering the most general $\mathcal{G}$-invariant symmetric two-tensor on $\mathcal{Y}$, we can define the metric on the coset space to be

$$g_{mn} e^m \otimes e^n := a(e^1 \otimes e^1 + e^2 \otimes e^2) + b(e^3 \otimes e^3 + e^4 \otimes e^4) + c(e^5 \otimes e^5 + e^6 \otimes e^6) \, , \quad (8.3.3)$$

where a, b, c are real. These three real parameters are the metric moduli of the space $\mathcal{Y}$, and we would like to relate them to the Kähler and Complex Structure forms. Our first step in doing this will be to construct specialisations of the two- and three-forms in (8.3.2) that obey (2.2.10), and will therefore be suitable for interpretation as the $SU(3)$-structure forms. Since some of the conditions of (8.3.2) involve the metric, constructing suitable forms also involves (8.3.3), and in fact uniquely determines the Kähler and Complex Structure forms in terms of a, b, c. A check on this procedure comes from (2.2.21). Imposing these constraints the $SU(3)$-structure forms are given by

$$\begin{aligned} J &= -ae^{12} + be^{34} - ce^{56} \, , \quad (8.3.4) \\ \Omega &= e^{i\varphi}\sqrt{abc}\left[\left(e^{135} + e^{146} - e^{236} + e^{245}\right) - i\left(e^{136} - e^{145} + e^{235} + e^{246}\right)\right] \, , \end{aligned}$$

where φ is an arbitrary phase which we can set to zero with no loss of generality, a choice that corresponds to choosing the torsion class conventions in (2.2.14). Now, since the basis on $\mathcal{Y}$ is just a subset of the generators of $\mathcal{G}$, their derivatives will be given in terms of the structure constants for $\mathcal{G}$, and provided the division by $\mathcal{H}$ has been performed adequately these derivatives should remain within $\mathcal{Y}$. Taking derivatives of the forms in (8.3.5) thus gives—as a specialisation of the result in [152]

$$\begin{aligned} dJ &= -(a+b+c)(e^{135}+e^{146}-e^{236}+e^{245}) \,, \\ d\Omega &= 4i\sqrt{abc}(e^{1256}-e^{1234}-e^{3456}) \,. \end{aligned} \tag{8.3.5}$$

Comparing (8.3.5) with (2.2.14), we see that $\mathcal{Y}$ belongs to the special class of half-flat manifolds defined in (8.2.16). Having found the appropriate forms and relations for J and Ω we can go on to look for a basis of expansion forms that satisfy (8.1.13) and (8.2.17). A consistent set of forms is given by

$$\omega_1 = -e^{12}, \ \omega_2 = e^{34}, \ \omega_3 = -e^{56} \,, \tag{8.3.6}$$

$$\widetilde{\omega}^1 = -e^{3456}, \ \widetilde{\omega}^2 = e^{1256}, \ \widetilde{\omega}^3 = -e^{1234} \,, \tag{8.3.7}$$

$$\alpha_0 = -e^{136}+e^{145}-e^{235}-e^{246} \,, \tag{8.3.8}$$

$$\beta^0 = -\frac{1}{4}\left(e^{135}+e^{146}-e^{236}+e^{245}\right) \,. \tag{8.3.9}$$

Note that we have made the choice $E_1 = E_2 = E_3 = 4$, however it would have been possible to choose different values for these parameters had we redefined the forms accordingly, and so this choice is simply for convenience. We have also chosen the normalisation convention $\int_{\mathcal{Y}} e^{123456} = 1$ so that the volume of $\mathcal{Y}$ is given by

$$\mathcal{V} = abc \,. \tag{8.3.10}$$

The structure forms J and Ω can be written in terms of this basis as

$$\begin{aligned} J &= a\omega_1 + b\omega_2 + c\omega_3 \,, \\ \Omega &= \sqrt{abc}\left(i\alpha_0 - 4\beta^0\right) \,. \end{aligned} \tag{8.3.11}$$

It is also worth noting that the torsion classes can be evaluated explicitly in this example, and are given by

$$\begin{aligned} \mathcal{W}_1 &= \frac{2i}{3}\frac{a+b+c}{\sqrt{abc}} \,, \\ \mathcal{W}_2 &= \frac{4i}{3}\frac{1}{\sqrt{abc}}\left[a(2a-b-c)e^{12} - b(2b-a-c)e^{34} + c(2c-a-b)e^{56}\right] \,. \end{aligned} \tag{8.3.12}$$

We have therefore been able to derive all the physically relevant quantities in terms of the real metric parameters a, b, c. We can now derive the effective theory arising from a compactification on the space $\mathcal{Y}$.

8.3.2 The effective theory

In section 8.3.1 above we showed that the space $\mathcal{Y}$ has three moduli associated with Kähler structure deformations. By comparing (8.1.33) with (8.3.11), we are able to relate them to the metric parameters

$$v^1 = a,\ v^2 = b,\ v^3 = c\,. \tag{8.3.13}$$

There were no geometric moduli associated with complex structure deformations. In the effective theory we therefore have three superfields T^1, T^2, T^3 from the Kähler structure sector and the superfield U^0 coming from the tensor multiplet. Using the decomposition of Ω^{cs} in (8.1.29), together with (8.1.31) and (8.3.11), gives $F_0 = -4iZ^0$, and so the superfields are

$$\begin{aligned} T^i &= b^i - iv^i\,, \\ U^0 &= \xi^0 + ie^{-\phi}\,. \end{aligned} \tag{8.3.14}$$

Our knowledge of the coset space also allows us to evaluate the superpotential (8.2.13) and the Kähler potential (8.2.8), which become

$$W = \frac{-i}{\sqrt{8}}\left[\lambda + 2mT^1T^2T^3 - 4\left(T^1+T^2+T^3\right)U^0\right]\,, \tag{8.3.15}$$

$$\mathcal{K} = -4\ln\left[-i\frac{1}{2}\left(U^0-\bar{U^0}\right)\right] - \ln\left[-i\left(T^1-\bar{T}^1\right)\left(T^2-\bar{T}^2\right)\left(T^3-\bar{T}^3\right)\right]\,. \tag{8.3.16}$$

We have now completely specified the $\mathcal{N}=1$ low energy effective theory on the space $\mathcal{Y}$. It is then natural to ask whether this theory has a stable vacuum. It is a well known result that supersymmetric minima are stable vacua. We therefore look for such a minimum by examining the F-term equations for the superpotential (8.3.15), which read

$$\begin{aligned} D_{T^1}W &= 2mT^2T^3 - 4U^0 - \frac{W}{T^1-\bar{T}^1} = 0\ , \\ D_{T^2}W &= 2mT^1T^3 - 4U^0 - \frac{W}{T^2-\bar{T}^2} = 0\ , \\ D_{T^3}W &= 2mT^1T^2 - 4U^0 - \frac{W}{T^3-\bar{T}^3} = 0\ , \\ D_{U^0}W &= -4\left(T^1+T^2+T^3\right) - \frac{4W}{U^0-\bar{U^0}} = 0\ , \end{aligned} \tag{8.3.17}$$

where the Kähler covariant derivative is given by $D_T := \partial_T + (\partial_T\mathcal{K})$. A solution to these equations can be found by setting $T^1 = T^2 = T^3 =: T$. In this case the equations simplify to the form

$$U^0 = \frac{1}{24T\overline{T}}\left(-T(\lambda+2m\overline{T}^3) + 3\overline{T}(\lambda+2mT^3)\right)\ , \tag{8.3.18}$$
$$0 = -6mT^2\overline{T}^3 - \lambda T\overline{T} - 2m\overline{T}T^4 + 3\lambda T^2 - 2\lambda\overline{T}^2 - 4mT^3\overline{T}^2 + 12mT\overline{T}^4\ .$$

A physically sensible solution to (8.3.18) should satisfy $m, e^{\mathcal{K}}, e^{-\phi} > 0$. Imposing these conditions gives a unique solution with $\lambda > 0$ where the vacuum expectation values for the superfield components are

$$\begin{aligned} \left\langle b^1\right\rangle = \left\langle b^2\right\rangle = \left\langle b^3\right\rangle &= -\frac{5^{\frac{2}{3}}}{20}\left(\frac{\lambda}{m}\right)^{\frac{1}{3}}\ , \\ \left\langle v^1\right\rangle = \left\langle v^2\right\rangle = \left\langle v^3\right\rangle &= \frac{\sqrt{3}5^{\frac{1}{6}}}{4}\left(\frac{\lambda}{m}\right)^{\frac{1}{3}}\ , \\ \left\langle \xi^0\right\rangle &= -\frac{5^{\frac{1}{3}}}{20}\left(m\lambda^2\right)^{\frac{1}{3}}\ , \\ \left\langle e^{-\phi}\right\rangle &= \frac{\sqrt{3}5^{\frac{5}{6}}}{20}\left(m\lambda^2\right)^{\frac{1}{3}}\ . \end{aligned} \tag{8.3.19}$$

It is easily shown that these values for the scalars satisfy the BCLT equations (8.2.1). The scalar potential is

$$V = e^{\mathcal{K}}\left[\mathcal{K}^{I\overline{J}}D_IWD_{\overline{J}}\overline{W} - 3\left|W\right|^2\right]\ , \tag{8.3.20}$$

where $I, J\ldots = 0,1,2,3$ label the superfields and $\mathcal{K}_{I\bar{J}} := \partial_I\partial_{\bar{J}}\mathcal{K}$ has inverse $\mathcal{K}^{I\bar{J}}$. Substituting (8.3.19), (8.3.16) and (8.3.15) into (8.3.20) we see that the cosmological constant in the vacuum is given by

$$\langle V \rangle = -3e^{\mathcal{K}}|W|^2 =: \Lambda \simeq \frac{-29.0}{(m\lambda^5)^{\frac{1}{3}}} , \tag{8.3.21}$$

and so the solution has an anti-de Sitter background. Having found a stable vacuum of the effective $\mathcal{N}=1$ theory the discussion in section 8.2.2 further implies that this is also a stable vacuum of the full $\mathcal{N}=2$ theory. The fact that it is a supersymmetric anti-de Sitter vacuum means that it is stable even if it is a saddle point [142, 143].

The moduli are therefore all stabilised without the use of any non-perturbative effects like instantons and gaugino condensation, or orientifold projections. To our knowledge this is the first example of such a vacuum. Because the stable vacuum arises from vevs of the scalar fields there is no freedom in choosing the flux parameters. The vacuum is in fact determined in terms of only two real parameters λ and m. This sits in contrast with the case of fluxes arising from branes, where the only handle on the generation of flux parameters comes from statistical 'landscape'-type considerations.

We may, however, eventually wish to consider uplifting the vacuum to a Minkowski or a de Sitter vacuum through a mechanism similar to the one used in the KKLT model [12]. Because such a possible uplift will most probably involve non-perturbative effects and new terms in the superpotential it may not leave the form of our solution unchanged. Nevertheless if an uplift leaves the solution unchanged the question of whether it is a full minimum or a saddle becomes important. We will therefore try to answer this question. We can construct a Hermitian block matrix from the second derivatives of the potential with respect to the superfields evaluated at the solution

$$H := \begin{pmatrix} V_{I\overline{J}} & V_{IJ} \\ V_{\overline{IJ}} & V_{J\overline{I}} \end{pmatrix} , \tag{8.3.22}$$

$$V_{I\overline{J}} = e^{\mathcal{K}}\mathcal{K}^{L\overline{M}}\partial_L\left(D_I W\right)\partial_{\overline{M}}\left(D_{\overline{J}}\overline{W}\right) - 2e^{\mathcal{K}}\mathcal{K}_{I\overline{J}}\left|W\right|^2 , \tag{8.3.23}$$

$$V_{IJ} = -\overline{W}e^{\mathcal{K}}\partial_I\left(D_J W\right) . \tag{8.3.24}$$

Then for the solution to be a local minimum in all the directions associated with the components of the superfields the matrix H must be positive definite. Inserting

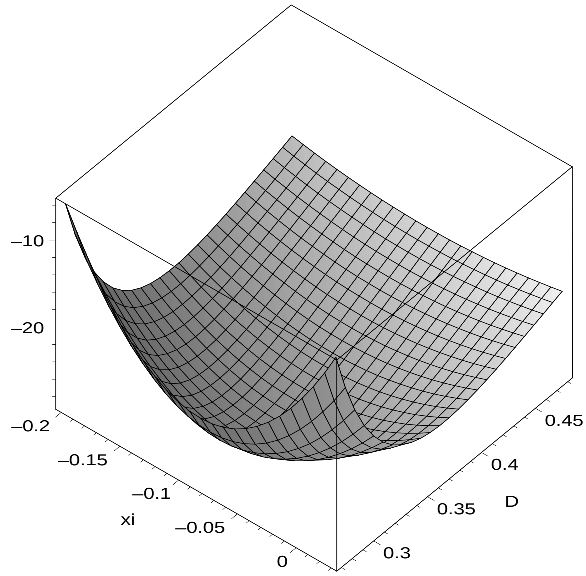

Figure 8.1: Plot showing the scalar potential for the directions ξ^0 and $e^{-\phi}$ (denoted as xi and D respectively).

the solution (8.3.19) into (8.3.24) we find that out of the eight real eigenvalues only six are positive. This means that there are two real directions for which the potential is at a maximum. We can determine these directions by looking at plots of the potential. Figure 8.1 shows the scalar potential for the two components of the U^0 (axio-dilaton) superfield at constant T^i with $\lambda = m = 1$. We see that the potential forms a minimum with respect these directions and so the maxima must be in directions associated with the T^i superfields. This raises the possibility that internal spaces with different geometrical structure to $\mathcal{Y}$ may evade this problem. To illustrate this we may consider the potential with the constraint $T^{1,2,3} =: \widetilde{T} =: \widetilde{b} - i\widetilde{v}$ imposed. This would correspond to an internal space with a single Kähler modulus, an example of which might be the coset $G_2/SU(3)$. Figure 8.2 shows the scalar potential for the directions associated with T at constant U^0. We see that again the potential forms a full minimum. Hence, although this is only an indication of how things might go, it provides motivation for the possibility of other spaces giving full minima and not saddles.

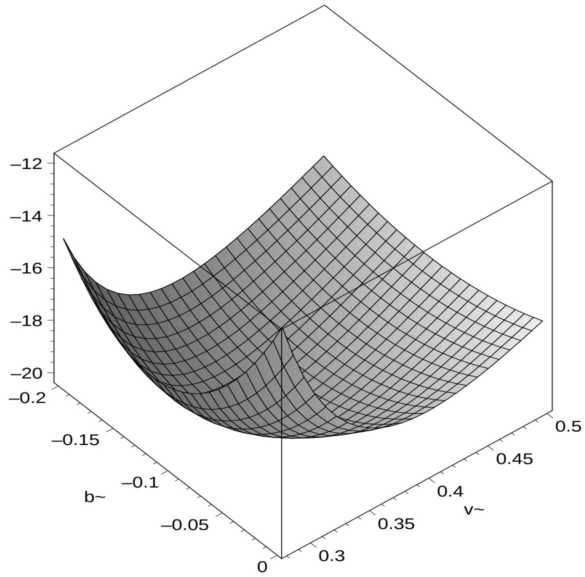

Figure 8.2: Plot showing the scalar potential for the directions $\widetilde{b}$ and $\widetilde{v}$ (denoted as b$\sim$ and v$\sim$ respectively).

Chapter 9

Conclusions

In this thesis, we have outlined some of the relatively new mathematics of G-structures as well as the more established concepts of special holonomy, both of which are relevant for the compactification of string- and M-theory. The use of torsion classes to classify supersymmetric supergravity solutions was shown, as was the relationship between G-structures and supersymmetry, and also the relationship between the G-structures, orientation and metric on a manifold. While the formalism of G-structures does not 'do the work' of solving either the equations of motion or the Killing spinor equations, they provide an elegant language in which to discuss such solutions.

We presented eleven-dimensional supergravity, which is both the dimensionally maximal supergravity and may be the low-energy limit of an eleven-dimensional quantum gravity obtained from the supermembrane worldvolume theory. This theory should be linked to the five known string theories by a set of dualities that we discussed. As well as membranes, a large number of different branes exist within string- and M-theory, and these play a crucial role in both considerations of non-perturbative formulations of string theory and in string- and M-theory phenomenology.

From the two main M-theory compactifications—Ricci-flat and Freund-Rubin—we selected the G_2 cases as the most interesting due to their preservation of $\mathcal{N} =$

1 supersymmetry in four dimensions. This amount of supersymmetry has many phenomenological advantages, including solution to the hierarchy problem and more readily obtained stable vacua, without the problems of higher supersymmetries such as lack of chiral representations.

There was a discussion in chapter 4 of both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity, with emphasis on the scalar sector. There was discussion of generic Kaluza-Klein reductions of fields in higher dimensions, together with discussion of many of the problems faced in string- and M-theory phenomenology. A variety of different solutions to these problems exist, with fluxes (sometimes via the branes that source these fluxes) playing an important role.

We went on to present some formal results about G_2, then analysed M-theory compactifications on manifolds with G_2 structure. Using the globally defined spinor which exists on such manifolds one can define the four-dimensional gravitino and compute explicitly the terms which give rise to the gravitino mass term in four dimensions. From this we were able to derive the general form for the superpotential which appears in M-theory compactifications on manifolds with G_2 structure (5.2.23). This formula generalises in a natural way the one that was derived for manifolds with G_2 holonomy in [15] which was derived based on the conjecture made in [83].

Even if a compact formula can be written for the superpotential, however, its expression in terms of the low energy fields is not known unless one specifies further the structure of the internal manifold. This was the purpose of chapter 6 where we derived the effective action that appears from compactifications of M-theory on manifolds with weak G_2 holonomy. It turns out that the possible metric variations on such manifolds are in one to one correspondence with the three-forms $\{\alpha\}$ on the weak G_2 manifold satisfying $d\alpha = -\tau \star \alpha$

Furthermore, fluctuations of the three-form field $\hat{A}_3$ that are proportional to such forms lead to scalar fields in four dimensions that are massless in the background AdS solution. Thus, as for the case of manifolds with G_2 holonomy, the metric fluctuations compatible with the structure and the massless modes of the three-form field $\hat{A}_3$ pair up into complex fields which will be the scalar components of the chiral superfields in four dimensions. The superpotential appears to be quadratic in the superfields (6.3.12) and we have shown by an explicit calculation that the potential which can be derived from this superpotential matches the one which appears from the compactification on the weak G_2 manifold.

In chapter 7, we studied M-theory compactifications on G_2-holonomy spaces in the presence of flux, both from the viewpoint of the eleven-dimensional theory and the associated four-dimensional supergravity theories. We solved the eleven-dimensional Killing spinor equations to linear order in flux and obtained G_2 domain walls, consisting of a warped product of a deformed G_2 space and a domain wall in four-dimensional space-time. The zero-mode parts of these solutions were reproduced, to all orders in flux, by solving the Killing spinor equations of the associated effective four-dimensional $\mathcal{N} = 1$ supergravity theories, obtained by reducing M-theory on G_2 spaces with flux. From this four-dimensional perspective, the solutions are domain walls which couple to the flux superpotential and with moduli varying non-trivially along the transverse direction. This transverse variation of the scalar fields can be seen as a path in the moduli space of G_2 metrics or, in other words, a variation of the internal G_2 space as one goes along the transverse direction.

We have also shown that these domain wall solutions can be sourced by either a membrane in four-dimensional space-time or an M5-brane wrapping a 3-cycle within the G_2 space. This leads to an interpretation of our solutions as the simplest manifestation of an M-theory 'topological defect' membrane or wrapped fivebrane appearing in a four-dimensional universe. We believe that studying such defects, arising from wrapped branes, in the context of M-theory cosmology is an interesting problem.

In chapter 8 we showed how the $\mathcal{N} = 2$ four-dimensional effective action for (massive) IIA supergravity on manifolds of $SU(3)$ structure can be constructed from the reduction of fermionic terms. We then went on to show that it is possible to break $\mathcal{N} = 2 \to \mathcal{N} = 1$ spontaneously by having the scalar fields pick up vevs. We derived the most general $\mathcal{N} = 1$ effective theory that can be obtained from such breaking.

Using an example manifold we showed how it is possible to stabilise all the fields in the vacuum without the use of any non-perturbative effects or orientifold projections. This is the first example we are aware of where moduli are stabilised in this manner. The real quantities λ, m are the only free parameters in our solution, which is far fewer than are usually required in moduli stabilisation.

9.1 Directions for future work

For all of the compactifications under consideration, it would be worth looking through (further) explicit examples of the manifolds under consideration to give some idea as to the possibilities for quantities derived in general.

The phenomenological viability of the models considered in chapters 6 and 8 is not very clear at the moment, mainly because of the large AdS curvatures of the four-dimensional spaces involved. Explicit examples of weak G_2 compactifications may, however, still be relevant in the process of obtaining the Standard Model spectrum from M-theory. It is well known that for chiral fermions to appear one needs conical singularities on the internal space [53, 54, 69]. However, until now there was no explicit construction of a *compact* manifold with G_2 holonomy which contains such singularities. On the other hand, the only explicit example of a compact manifold with G_2 structure which has such singularities is the case constructed in [72], which is a weak G_2 manifold. Considering an explicit example of such a space would be interesting both in terms of looking for supersymmetric minima and BPS states in that model and also for studying anomaly cancellation.

A further task for looking at phenomenology from these models would be to look at getting a realistic physics either through the use of intersecting branes or through the use of the A-D-E singularities needed to obtain gauge bosons.

It would also be of interest, for models where the moduli are stabilised, to study the cosmology of the scalars as they roll towards the vacuum. There are also the questions discussed earlier in this paper as to whether the inclusion of non-perturbative effects could lift the vacuum to a de Sitter background.

Our four-dimensional domain walls diverge away from the wall and, in particular, do not approach Minkowski space. This is a common feature of supergravity domain walls [135]. It would be interesting to see how these solutions are modified by the inclusion of a non-perturbative superpotential in four dimensions and whether this can remove the divergences.

Furthermore there is the possibility of paying more attention to BPS states in the scenarios, in particular the cosmic string, which should appear in the M-theory case from wrapping the fivebrane on a co-associative cycle on the G_2 holonomy space, in the same way that a domain wall appeared from wrapping the fivebrane on an associative cycle on the G_2 holonomy space.

There are several other less immediately phenomenological directions which are worth investigating. The first of these would be a systematic study of the mass operators (6.3.20) in AdS backgrounds, which will be more complicated for the (massive) IIA case, and which may also admit stable states of mass$^2 \leq 0$ that we have not considered here.

Another open question is whether there are any systematic ways to study the moduli spaces of G-structure manifolds that would determine whether our assumptions and results about the basis for Kaluza-Klein reduction can be proved or extended for a more general case.

Although the results of chapter 8 depend on some specific features of the massive IIA supergravity, it may be possible to obtain similar $\mathcal{N} = 2 \to \mathcal{N} = 1$ spontaneous breaking for other theories, for example IIB and M-theory on manifolds of $SU(3)$ structure or Type I and Heterotic string theories on manifolds of $SU(2)$ structure. These manifolds offer several globally defined forms in terms of which vev-derived fluxes could be written that might drive the super-Higgs mechanism.

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Conventions

Here we outline the conventions that we use throughout this work.

Indices

Although it is almost impossible to avoid some use of the same type of index for different purposes, we list here the most common uses of each type of index, in the rough order of use.

$a, b \ldots$	Generic indices; subset of seven-dimensional spacetime indices; non-zero complex-strucutre moduli-space indices; gauge indices.
$A, B \ldots$	Seven-dimensional spacetime indices; complex-strucutre moduli-space indices; spinor labels.
$i, j \ldots$	G_2 moduli space indices; Kähler moduli-space indices; brane worldvolume indices.
$I, J \ldots$	Eleven-dimensional spacetime indices; Kähler and complex-strucutre moduli-space indices;
$m, n \ldots$	Six-dimensional spacetime indices; D-dimensional spacetime indices
$M, N \ldots$	Ten-dimensional spacetime indices; $(4 + D)$-dimensional spacetime indices
$\alpha, \beta \ldots$	$SU(2)$ indices; holomorphic indices
$\overline{\alpha}, \overline{\beta} \ldots$	anti-holomorphic indices
$\mu, \nu \ldots$	Four-dimensional spacetime indices; three-dimensional spacetime indices; generic spacetime indices.

Underlined indices denote directions in the corresponding tangent space.

Metric and volume element

Throughout this work we have used the space-time metric signature $(-,+,+,\ldots)$. We define the ϵ symbol such that $\hat{\epsilon}_{0123..} := +1$ with

$$\epsilon_{01\ldots D-1} = +\sqrt{|\det(g)|} \ . \tag{9.1.1}$$

Indices are raised and lowered with the metric. We have used g throughout to denote various metrics—which one is meant can be determined from the indices used or from context.

Forms

The relation between index-free and index notation is given by

$$A_{(p)} = \frac{1}{p!} A_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \ . \tag{9.1.2}$$

In our conventions, the standard operations on forms in D dimensions are given by

$$\begin{aligned}
(dA)_{\mu_1\ldots\mu_{p+1}} &= (p+1)\nabla_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]} \ , \\
(d^\dagger A)_{\mu_1\ldots\mu_{p-1}} &= -\nabla^\nu A_{\nu\mu_1\ldots\mu_{p-1}} \ , \\
(\star A)_{\mu_1\ldots\mu_{D-p}} &= \frac{(-1)^{p(D-p)}}{p!} \epsilon_{\mu_1\ldots\mu_{D-p}}{}^{\nu_1\ldots\nu_p} A_{\nu_1\ldots\nu_p} \ .
\end{aligned} \tag{9.1.3}$$

Furthermore, $\lrcorner$ denotes the contraction of indices, so that for a p-form ω_p and a $q \geq p$ form Ω_q

$$(\omega_p \lrcorner \Omega_q)_{\mu_1\ldots\mu_{q-p}} = (\omega_p)^{\nu_1\ldots\nu_p} (\Omega_q)_{\nu_1\ldots\nu_p\mu_1\ldots\mu_{q-p}} \ . \tag{9.1.4}$$

We also use the symbol '$\cdot$' for this contraction when carried out over four-dimensional Minkowskian indices.

Dirac matrices

General properties

Generic Dirac matrices $\{\Gamma^{\mu}\}_{\mu=1}^{d}$ are defined to obey

$$\{\Gamma^{\underline{\mu}}, \Gamma^{\underline{\nu}}\} = \begin{cases} \delta^{\underline{\mu\nu}} & \text{for Euclidian signature,} \\ \eta^{\underline{\mu\nu}} & \text{for Minkowskian signature.} \end{cases} \tag{9.1.5}$$

The curved-space Dirac matrices then obey

$$\Gamma^{\mu} = e^{\mu}_{\underline{\mu}} \Gamma^{\underline{\mu}} \quad \Rightarrow \quad \{\Gamma^{\mu}, \Gamma^{\nu}\} = g^{\mu\nu} \ , \tag{9.1.6}$$

where $e^{\mu}_{\underline{\mu}}$ is the d-dimensional vielbein. Multiple indices are defined to mean

$$\Gamma^{\mu_1 \cdots \mu_p} = \Gamma^{[\mu_1 \cdots} \Gamma^{\mu_p]} \ . \tag{9.1.7}$$

For the rest of this section, we will work in flat space, and will give conventions for the various sets of Dirac matrices used.

Three dimensions

In our conventions, the 3-dimensional Minkowskian Dirac matrices are given by

$$\begin{aligned} \rho^0 &= -i\sigma^2 \ , \\ \rho^1 &= \sigma^1 \ , \\ \rho^2 &= -\sigma^3 \ . \end{aligned} \tag{9.1.8}$$

These obey the following

$$\rho^0 \rho^1 \rho^2 = -\mathbb{1}_2 \ , \tag{9.1.9}$$

$$(\rho^{\mu})^* = \rho^{\mu} \ , \tag{9.1.10}$$

$$\{\rho^{\mu}, \rho^{\nu}\} = 2\eta^{\mu\nu} \ . \tag{9.1.11}$$

Four dimensions

The four-dimensional Minkowskian Dirac matrices are constructed from the above by

$$\begin{aligned} \gamma^\mu &= \rho^\mu \otimes \sigma^1 \, , \\ \gamma^y &= \mathbb{1}_2 \otimes \sigma^2 \, . \end{aligned} \tag{9.1.12}$$

We can define a 4D chirality operator $\gamma = i\gamma^0\gamma^1\gamma^2\gamma^y$ so that

$$\begin{aligned} \{\gamma, \gamma^y\} = \{\gamma, \gamma^\mu\} = \{\gamma^\mu, \gamma^y\} &= 0 \, , \\ \{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu} \, , \\ (\gamma)^2 = (\gamma^y)^2 &= \mathbb{1} \, . \end{aligned} \tag{9.1.13}$$

These matrices further obey the following

$$\hat{\varepsilon}_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho = 6i\gamma^y\gamma \, , \tag{9.1.14}$$

$$\hat{\varepsilon}^\mu{}_{\nu\rho}\gamma^\nu\gamma^\rho = 2i\gamma^y\gamma\gamma^\mu \, , \tag{9.1.15}$$

$$(\gamma^\mu)^* = \gamma^\mu \, , \quad (\gamma)^* = \gamma \, , \quad (\gamma^y)^* = -\gamma^y \quad . \tag{9.1.16}$$

Six dimensions

In six-dimensional Euclidean space it is possible to define a set of Dirac matrices $\{\gamma^m\}_{m=1\ldots6}$ that are purely imaginary so that

$$(\gamma^m)^* = -\gamma^m \, , \tag{9.1.17}$$

$$\{\gamma^m, \gamma^n\} = 2\delta^{mn} \, . \tag{9.1.18}$$

Seven dimensions

In seven-dimensional Euclidean space it is possible to define a set of Dirac matrices $\{\gamma^A\}_{A=1\ldots7}$ that are purely imaginary so that

$$(\gamma^A)^* = -\gamma^A \, , \tag{9.1.19}$$

$$\{\gamma^A, \gamma^B\} = 2\delta^{AB} \, . \tag{9.1.20}$$

Clearly, the seventh matrix can be written as a product

$$\gamma^7 = i\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6 \ . \tag{9.1.21}$$

γ^7 is also the six-dimensional chirality operator.

Ten dimensions

We define the set of ten-dimensional Dirac matrices

$$\left\{\Gamma^I\right\}_{I=0\ldots9} = \{\Gamma^\mu, \Gamma^m\}_{\mu=0\ldots3, A=1\ldots6} \ , \tag{9.1.22}$$

by the relations

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \gamma^7 \ , \\ \Gamma^m &= \gamma \otimes \gamma^m \ . \end{aligned} \tag{9.1.23}$$

Eleven dimensions

Finally, we define the set of eleven-dimensional Dirac matrices

$$\left\{\Gamma^I\right\}_{I=0\ldots10} = \left\{\Gamma^\mu, \Gamma^A\right\}_{\mu=0\ldots3, A=1\ldots7} \ , \tag{9.1.24}$$

by the relations

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathbb{1}_8 \ , \\ \Gamma^A &= \gamma \otimes \gamma^A \ . \end{aligned} \tag{9.1.25}$$

Spinors

$\overline{\psi}$ conjugation

Our conventions for spinor conjugation are that, for general spinor ψ, in Minkowskian-signature spaces we have

$$\overline{\psi} := \psi^\dagger\gamma^0 \ , \tag{9.1.26}$$

where † denotes Hermitian conjugation, and in Euclidean-signature spaces

$$\overline{\psi} := \psi^\dagger \ . \tag{9.1.27}$$

Majorana conjugation

For a general spinor ψ, we define its Majorana conjugate in terms of the matrix B

$$\psi^c = B^{-1}\psi^* \ . \tag{9.1.28}$$

Imposing that this operation should commute with Lorentz transformations and square to unity gives the conditions

$$B\Gamma^I B^{-1} = \pm(\Gamma^I)^* \ , \qquad B^* B = \mathbb{1} \ . \tag{9.1.29}$$

We are interested in imposing the Majorana condition mainly in eleven-dimensional compactifications to four dimensions. Let us therefore decompose B_{11} as $B_{11}^{-1} = B_4^{-1} \otimes B_7^{-1}$ into four- and a seven-dimensional conjugation matrices B_4 and B_7. They must satisfy the relations

$$B_4\gamma^\mu B_4^{-1} = \pm(\gamma^\mu)^* \ , \qquad B_7\gamma^A B_7^{-1} = \mp(\gamma^A)^* \ , \tag{9.1.30}$$

where $\mu = 0 \ldots 3$, in order to reproduce Eq. (9.1.29). In our conventions, these matrices can be represented as

$$B_4^{-1} = \gamma^y\gamma \quad B_7^{-1} = \mathbb{1} \ . \tag{9.1.31}$$